

DEPENDENT FIRST ORDER THEORIES, CONTINUED

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ABSTRACT. A dependent theory is a (first order complete theory) T which does not have the independence property. A main result here is: if we expand a model of T by the traces on it of sets definable in a bigger model then we preserve its being dependent. Another one justifies the cofinality restriction in the theorem (from a previous work) saying that pairwise perpendicular indiscernible sequences, can have arbitrary dual-cofinalities in some models containing them.

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ANNOTATED CONTENT

Recall : dependent $T = T$ without the independence property.

§0 Introduction

§1 Expanding by making a type definable, p.4

[Suppose we expand $M \prec \mathfrak{C}$ by a relation for each set of the form $\{\bar{b} : \bar{b} \in {}^m M \text{ and } \models \varphi[\bar{b}, \bar{a}]\}$, where $\bar{a} \in {}^{\omega>} \mathfrak{C}$, $\varphi(\bar{x}, \bar{y}) \in \mathbb{L}(\tau_T)$ and $m = \ell g(\bar{x})$. We prove that the theory of this model is dependent and has elimination of quantifiers.]

§2 More on indiscernible sequences, p.11

[complimentary to ([Sh 715, §5]); cuts with cofinality from both sides $< \kappa + |T| = \kappa$ inside κ -saturated models tend to be filled together; question:

- (a) small cofinality in one side
- (b) expand as indiscernible set types of pairs is a partial order?]

§3 Strongly dependent theories, p.20

$[(1 = \aleph_0)$ -type, contradict strongly dependent; f definable function; question: regular types even for stable?]

§4 Definable groups, p.25

[if $B \subseteq Q$ is commuting then T is dependent. If each x commutes with finitely many.]

§5 Non forking, p.34

§0 INTRODUCTION

The work in [Sh 715] tries to deal with the investigation of (first order complete) theories T which has the dependence property, i.e., does not have the independence property.

If T is stable expanding a model M of T by $p \restriction \varphi(\bar{x}, \bar{y})$ for $p \in \mathcal{S}^m(M)$, i.e., by the relation $R_{p,M}^{\varphi(\bar{x}, \bar{y})} = \{\bar{a} \in {}^{\ell g(\bar{y})}M : \varphi(\bar{x}, \bar{a}) \in p\}$ is an inessential one, i.e., by a relation definable with parameters. This fails for unstable theories but in §1 we prove a weak relative: if T is a dependent theory then so is the expansion above, i.e., $\text{Th}(M, R_{p,M}^{\varphi(\bar{x}, \bar{y})})$.

In [Sh 715, §5] it is shown that we can construct a model M of a dependent unstable T , we can find κ -saturated model M such that the following set is quite arbitrary: the set $\text{cut}(M)$ of pairs of partial orders in M (so not fulfilled in M); or even: there is an indiscernible sequence $\langle a_\alpha : \alpha < \kappa_1 \rangle \frown \langle b_\alpha : \alpha < \kappa_2^* \rangle$ such that the (κ_1, κ_2^*) -cut is respected in M . However, there were restrictions on the cofinalities being not too small. Are those restrictions necessary? In §2 we show that to a large extent, yes.

The family of dependent theories is parallel to the family of stable theories. But actually a better balance of “size” of the family of such theories and what we can tell about them is obtained by the family of superstable ones. In §3 a relative, strongly dependent theories, is defined. In fact, even for stable theories there are fewer (see also [Sh 839], [Sh:F660]). We then observe some basic properties. This is continued in [Sh:F671].

In §4 we look at groups definable in models of dependents, and more strongly dependent theories. In §5 we try to look systematically parallel to non-forking.

Notation:

As in [Sh 715] and, in addition

0.1 Definition. 1) For $\bar{\mathbf{b}} = \langle \bar{b}_t : t \in I \rangle$ an infinite indiscernible sequence, let $\text{tp}'(\bar{\mathbf{b}}) = \langle \text{tp}(\bar{b}_{t_0^n} \hat{\ } \dots \bar{b}_{t_{n-1}^n}, \emptyset, \mathfrak{C}) : n < \omega \rangle$ where $t_\ell^n <_I t_{\ell,k}^n$ for $\ell < k < n < \omega$, their choice is immaterial.

2) Let “ M is n -saturated” mean “ M is \aleph_0 -saturated” for $n < \omega$.

3) A/B is $\text{tp}(A, B)$, inside \mathfrak{C} or \mathfrak{C}^{eq} , of course.

§1 EXPANDING BY MAKING A TYPE DEFINABLE

- 1.1 *Content.* 1) T is a (first order complete) theory in the language $\mathbb{L}(T)$.
 2) $\mathfrak{C} = \mathfrak{C}_T$ is a monster model for T .

1.2 **Claim.** *Assume*

- (a) M a model
- (b) D an ultrafilter on M , i.e. on the Boolean Algebra $\mathcal{P}(M)$
- (c) $\text{def}(D) = \{A \in D: \text{for some } \bar{c} \in {}^\omega \mathfrak{C} \text{ and formula } \psi(x, \bar{c}) \text{ the set } \{a \in M : \mathfrak{C} \models \psi(a, \bar{c})\} \text{ belong to } D \text{ and is included in } A\}$.

Then for any $\bar{c} \in {}^\omega \mathfrak{C}$ and formula $\varphi(x, y, \bar{c})$ we have: if the set $\{a \in M : (\exists y \in M)(\mathfrak{C} \models \varphi[a, y, \bar{c}])\}$ belongs to D then it belongs to $\text{def}(D)$.

1.3 *Remark.* Of course, this holds also for $\varphi = \varphi(\bar{x}, \bar{y}, \bar{c})$ when D an ultrafilter on ${}^m M$ and $m = \ell g(\bar{y})$ (why? e.g. just you should work in \mathfrak{C}^{eq}).

Proof. Assume toward contradiction that $\bar{c}, \varphi(x, y, \bar{c})$ form a counterexample. So the set $A^* = \{a \in M: \text{for some } b \in M \text{ we have } \models \varphi[a, b, \bar{c}]\}$ belong to D , and for each $a \in A^*$ choose $b_a \in M$ such that $\models \varphi[a, b_a, \bar{c}]$. Let $D_1 = D$ and let D_2 be the following ultrafilter on ${}^2 M$: $X \in D_2$ iff $X \subseteq {}^2 M$ and for some $A \in D$ we have $\{(a, b_a) : a \in A \cap A^*\} \in D$.

We can choose $\langle (a_{\omega+n}, b_{\omega+n}) : n < \omega \rangle$ from \mathfrak{C} such that for $n_1 < n_2 < \omega$ the pair $(a_{\omega+n_1}, b_{\omega+n_1})$ realizes the type $\text{Av}(M \cup \cup \{a_{\omega+\ell}, b_{\omega+\ell} : \ell \in (n_1, n_2]\}, D_2)$. Now clearly

- \boxtimes_0 $\langle (a_{\omega+n}, b_{\omega+n}) : n < \omega \rangle$ is an indiscernible sequence over M
- \boxtimes_1 if $n_1 < \omega$ and $\Delta_1 \subseteq \mathbb{L}(T)$ is finite then we can find $n_2 < \omega$ and finite $\Delta_2 \subseteq \mathbb{L}(T)$ such that
 - $(*)_1$ if for each $\ell < n_1, a_\ell \in M$ realizes $\text{tp}_{\Delta_2}(a_\omega, \{a_0, \dots, a_{\ell-1}\} \cup \{a_{\omega+1}, \dots, a_{\omega+n_2}\})$ then $\langle a_\ell : \ell < n_1 \rangle^\wedge \langle a_{\omega+\ell} : \ell < n_1 \text{ and } \ell \geq 1 \rangle$ is a Δ_1 -indiscernible sequence.

[Why? Straight by induction on the arity of Δ_1 (or quote [Sh 715, §1]); in fact $n_1 = \omega$ is O.K.]
- \boxtimes_2 if $n_1 < \omega_1$ and $\Delta_1 \subseteq \mathbb{L}(\tau_T)$ is finite then we can find $n_2 < \omega$ and finite $\Delta_2 \subseteq \mathbb{L}(T)$ such that
 - $(*)_2$ if for each $\ell < n_1, a_{2\ell} \in M$ realizes $\text{tp}_{\Delta_2}(a_\omega, \{a_{2m}, a_{2m+1}, b_{2m+1} : m < \ell\} \cup \{a_{\omega+\ell}, b_{\omega+\ell} : \ell = 1, \dots, n_2\})$

and $\langle a_{2\ell+1}, b_{2\ell+1} \rangle$ realizes $\text{tp}_{\Delta_2}((a_\omega, b_\omega), \{a_{2m}, a_{2m+1}, b_{2m+1} : m < \ell\} \cup \{a_{2\ell}\} \cup \{a_{\omega+\ell}, b_{\omega+\ell} : \ell = 1, \dots, n_2\})$ then $\langle (a_{2\ell}, a_{2\ell+1}, b_{2\ell+1}) : \ell < n_1 \text{ and } \ell \geq 1 \rangle \wedge \langle a_{\omega+2\ell}, a_{\omega+2\ell+1}, b_{\omega+2\ell+1} : \ell < n_1 \rangle$ is Δ_1 -indiscernible.

[Why? By induction on the arity of Δ_1 (or quote [Sh 715, §1]).]

- ⊠₃ if $B \subseteq M$ is finite, $n^* < \omega$ and $\Delta \subseteq \mathbb{L}(\tau_T)$ is finite, then we can find $a \in M$ realizing the finite type $q = \text{tp}_\Delta(a_\omega, B \cup \{a_{\omega+\ell}, b_{\omega+\ell} : \ell = 1, \dots, n^*\})$ such that $\models \neg(\exists y \in M)\varphi(a, y, \bar{c})$

[Why? The set $A =: \{a \in M : a \text{ realizes } q, \text{ equivalently satisfies the formula } \wedge q \in \text{Av}(\mathfrak{C}, D)\}$ belong to D because q is finite and the choice of D_2 and $\langle a_{\omega+\ell}, b_{\omega+\ell} : \ell < \omega \rangle$; moreover, it belongs to $\text{def}(D)$ by the definition of $\text{def}(D)$ and choice of A . So by the assumption toward contradiction, $\neg(A \subseteq A^*)$ so there is $a \in A$ such that $\neg(\exists y \in M)\varphi(a, y, \bar{c})$, so we are done.]

By the above and compactness (or use an ultrapower)

- ⊠₄ there are $N, a_{2n}, a_{2n+1}, b_{2n+1}$ (for $n < \omega$) such that
- (a) N is $|T|^+$ -saturated
 - (b) $a_{2n}, a_{2n+1}, b_{2n+1} \in N$
 - (c) $\langle a_n : n < \omega \rangle$ is an indiscernible sequence
 - (d) $\langle (a_{2n}, a_{2n+1}, b_{2n+1}) : n < \omega \rangle \wedge \langle (a_{\omega+2n}, a_{\omega+2n+1}, b_{\omega+2n+1}) : n < \omega \rangle$ is an indiscernible sequence
 - (e) $\models \varphi[a_{2n+1}, b_{2n+1}, \bar{c}]$
 - (f) for no $n < \omega$ and $b \in N$ do we have $\models \varphi[a_{2n}, b, \bar{c}]$.

Next easily

- ⊠₅ there is an automorphism F of \mathfrak{C} such that
- $n < \omega$ implies $F((a_{2n}, a_{2n+1}, b_{2n+1}) = (a_{4n+1}, a_{4n+3}, b_{4n+3})$ and $F(a_{\omega+n}) = a_{\omega+n}$.

Hence we can find $b_{2n} \in \mathfrak{C}$ for $n < \omega$ such that $\langle (a_n, b_n) : n < \omega \rangle$ is an indiscernible sequence and as N is $|T|^+$ -saturated without loss of generality $b_{2n} \in N$ for $n < \omega$. But $\models \varphi[a_{2n+1}, b_{2n+1}, \bar{c}]$ for $n < \omega$ so as T is dependent for every large enough n , $\models \varphi[a_{2n}, b_{2n}, \bar{c}]$ but as $b_{2n} \in N$ this contradicts clause (f) of ⊠₄. $\square_{1.2}$

1.4 Conclusion. 1) Assume

- (a) $M \subseteq C$

- (b) D^0 is an ultrafilter on ${}^{m_0}M$
- (c) \bar{b}_0 realizes $\text{Av}(C, D_0)$
- (d) $\text{tp}(\bar{b}_0 \hat{\ } \bar{b}_1, C)$ is f.s. in M and $m_1 = \ell g(\bar{b}_1)$
- (e) C is full over M (which means that every $p \in \mathcal{S}^{<\omega}(M)$ is realized in C).

Then for some ultrafilter D on ${}^{m_0+m_1}M$ we have

- (α) $\text{Av}(C, D) = \text{tp}(\bar{b}_0 \hat{\ } \bar{b}_1, C)$
- (β) the projection of D on ${}^{m_0}M$ is D_0 .

2) Assume that clauses (a) and (e) of part (1) holds. Then for any $\bar{c} \in {}^{\omega>}\mathfrak{C}$ and formula $\varphi(x, y, \bar{z}) \in \mathbb{L}(T)$, $\ell g(\bar{z}) = \ell g(\bar{c})$ there are $\psi(x, \bar{z}')$ and \bar{d} of length $\ell g(\bar{z}')$ from \mathfrak{C} , and even from C such that $\{a \in M : (\exists y \in M)(\models \varphi[a, y, \bar{c}])\} = \{a \in M : \models \psi(a, \bar{d})\}$.

3) If

- (a) D_1, D_2 are ultrafilters on ${}^m A$,
- (b) $A \subseteq C$,
- (c) C full over A , that is every $p \in \mathcal{S}^{<\omega}(A)$ is realized by some sequence from C
- (d) $\text{Av}(C, D_1) = \text{Av}(C, D_2)$.

Then $\text{def}(D_1) = \text{def}(D_2)$.

Proof. 1) Let

$$\mathcal{E}_0 = \{\{\bar{a} \in {}^{m_0+m_1}M : \bar{a} \upharpoonright m_0 \in X\} : X \in D_0\}$$

$$\begin{aligned} \mathcal{E}_1 = \{ \{ \bar{a} \in {}^{m_0+m_1}M : \models \varphi[\bar{a}; \bar{c}] \} : & \varphi(\bar{x}; \bar{y}) \in \mathbb{L}(\tau_T) \\ & \ell g(\bar{x}) = m_0 + m_1, \ell g(\bar{y}) = \ell g(\bar{c}), \\ & \bar{c} \subseteq C \text{ and } \models \varphi[\bar{b}_0 \hat{\ } \bar{b}_1; \bar{c}] \}. \end{aligned}$$

Clearly it suffices to prove that there is an ultrafilter on ${}^{m_0+m_1}M$ extending $\mathcal{E}_0 \cup \mathcal{E}_1$. For this it suffices to show that for any finite subfamily of $\mathcal{E}_0 \cup \mathcal{E}_1$ has a non empty intersection. But \mathcal{E}_0 is closed under finite intersections as D^0 is an ultrafilter on ${}^{m_0}M$ and \mathcal{E}_1 is closed under finite intersections as $\mathbb{L}(\tau_T)$ is closed under conjunctions, so it suffices to prove that $X_0 \cap X_1 \neq \emptyset$ when

- (i) $X_0 = \{\bar{a} \in {}^{m_0+m_1}M : \bar{a} \upharpoonright m_0 \in X\} \in \mathcal{E}_0$ for some $X \in D_0$
- (ii) $X_1 = \{\bar{a} \in {}^{m_0+m_1}M : \models \varphi[\bar{b}_0 \bar{\wedge} \bar{b}_1; \bar{c}]\} \in \mathcal{E}_1$ where $\varphi(\bar{x}, \bar{y}), \bar{c}$ are as in the definition of \mathcal{E}_1 .

Let D'_1 be an ultrafilter on ${}^{m_0+m_1}M$ such that $\text{Av}(C, D^1) = \text{tp}(\bar{b}_0 \bar{\wedge} \bar{b}_1, C)$, as $\text{tp}(\bar{b}_0 \bar{\wedge} \bar{b}_1, C)$ is finitely satisfiable in M (= assumption (d)) clearly such D'_1 exists and let D'_0 be the projection of D'_1 to ${}^{m_0}M$, i.e., $\{Y \subseteq {}^{m_0}M : \{\bar{a} \in {}^{m_0+m_1}M : \bar{a} \upharpoonright m_0 \in Y\} \in D'_1\}$. As $\models \varphi[\bar{b}_0, \bar{b}_1; \bar{c}]$, clearly $X_1 \in D_1$ hence $X'_0 = \{\bar{a} \upharpoonright m_0 : \bar{a} \in X_1\} \in D'_0$; which implies that $X''_0 =: \{\bar{a}_0 \in {}^{m_0}M : \text{for some } \bar{a}_2 \in {}^{m_1}M \text{ we have } \bar{a}_0 \bar{\wedge} \bar{a}_1 \in X_1, \text{ i.e., } \models \varphi[\bar{a}_0, \bar{a}_1; \bar{c}]\} \in D'_0$.

By 1.2 it follows that $X''_0 \in \text{def}(D'_0)$. But as $\text{Av}(C, D_0) = \text{Av}(C, D'_0)$ being $\text{tp}(\bar{b}_0, C)$ and as by assumption (e) every $p \in \mathcal{S}^{<\omega}(M)$ is realized by some sequence from C , by part (3) below we have $\text{def}(D_0) = \text{def}(D'_0)$. Hence $X''_0 \in \text{def}(D_0)$ so $X''_0 \in D_0$, but also $X \in D_0$ hence we can find $\bar{a}_0 \in X \cap X''_0 \subseteq {}^{m_0}M$. By the definition of X''_0 there is $\bar{a}_1 \in {}^{m_1}M$ such that $\models \varphi[\bar{a}_0, \bar{a}_1; \bar{c}]$. hence $\bar{a}_0 \bar{\wedge} \bar{a}_1 \in X_1$ by X_1 definition, that is, clause (ii) and $\bar{a}_0 \bar{\wedge} \bar{a}_1 \in X_0$ by the X_0 's definition, i.e., clause (i) as $\bar{a}_0 \bar{\wedge} \bar{a}_1 \in X_0 \cap X_1$ so $X_0 \cap X_1 \neq \emptyset$ and we are done.

2) Let $Y_{\psi(x, \bar{c}), M} = \{a \in M : \models \psi[a, \bar{c}]\}$ and $\{X_{\varphi(x, y, \bar{c}), M} = \{a \in M : \models \varphi[b, \bar{c}]\}$ for some $b \in M\}$.

Lastly, let $\mathcal{P} = \{Y_{\psi(x, \bar{d}), M} : \psi(x, \bar{y}) \in \mathbb{L}(\tau_T), \bar{d} \in {}^{\ell g(\bar{z})}\mathfrak{C} \text{ and } Y_{\psi(x, \bar{d}), M} \subseteq X_{\varphi(x, y, \bar{c}), M}\}$.

Clearly \mathcal{P} is closed under finite unions and is a family of subsets of $X_{\varphi(x, y, \bar{c}), M}$. Also if $X_{\varphi(x, y, \bar{c}), M}$ belongs to \mathcal{P} then we are done, so assume toward contradiction that this fails, and so as $X_{\varphi(x, y, \bar{c}), M} \subseteq M$ there is an ultrafilter D on M such that $X_{\varphi(x, y, \bar{c}), M} \in D$ but D is disjoint to \mathcal{P} . This contradicts 1.2.

3) Trivial. □_{1.4}

1.5 Conclusion. Assume

- (a) $M \prec M_1$
- (b) M_1 is $\|M\|^{+}$ -saturated.

Then $\{A : A/M_1 \text{ is f.s. in } M\}$ has amalgamation (and joint embedding property).

Proof. The joint embedding property is trivial. For the amalgamation, by compactness we should consider finite sequence $\bar{a}_0, \bar{a}_1, \bar{a}_2$ such that $\text{tp}(\bar{a}_0 \bar{\wedge} \bar{a}_\ell, M_1)$ is f.s. in M_0 for $\ell = 1, 2$ and we should find sequences $\bar{b}_0, \bar{b}_1, \bar{b}_2$ such that $\ell g(\bar{b}_\ell) = \ell g(\bar{a}_\ell)$ and $\text{tp}(\bar{a}_0 \bar{\wedge} \bar{a}_\ell, M_1) = \text{tp}(\bar{b}_0 \bar{\wedge} \bar{b}_\ell, M_1)$ for $\ell = 1, 2$ and $\text{tp}(\bar{b}_0 \bar{\wedge} \bar{b}_1 \bar{\wedge} \bar{b}_2, M_1)$ is f.s. in M . Let $m_\ell = \ell g(a_\ell)$, let D_0 be an ultrafilter on ${}^{m_0}M$ such that $\text{tp}(\bar{a}_0, M_1) = \text{Av}(M_1, D_0)$. By 1.4(1) for $\ell \in \{1, 2\}$ there is an ultrafilter D_ℓ on ${}^{m_0+m_\ell}M$ such that

- (*)₁ $\text{tp}(\bar{a}_0 \hat{\ } \bar{a}_\ell, M_2)$ is $\text{Av}(M_2, D_\ell)$
- (*)₂ the projection of D_ℓ on ${}^{m_0}M$ is D_0 .

Let $m = m_0 + m_1 + m_2$ and let D'_1 be the filter on mM consisting of $\{Y \subseteq {}^mM : \text{for some } X \in D_1 \text{ for every } \bar{a} \in {}^mM \text{ we have } \bar{a} \upharpoonright (m_0 + m_1) \in X \Rightarrow \bar{a} \in Y\}$ and let D'_2 be the filter on mM consisting of $\{Y \subseteq {}^mM : \text{for some } \bar{a} \in {}^mM \text{ we have } (\bar{a} \upharpoonright m_0) \hat{\ } (\bar{a} \upharpoonright [m_0 + m_1, m]) \in X \Rightarrow \bar{a} \in Y\}$. Easily $Y_1 \in D'_1$ & $Y_2 \in D'_2 \Rightarrow Y_1 \cap Y_2 \neq \emptyset$.

Now we can find an ultrafilter D^* on ${}^{m_0+m_1+m_2}M$ such that it extends $D'_1 \cup D'_2$ hence if $\bar{b}_0 \hat{\ } \bar{b}_1 \hat{\ } \bar{b}_2$ realizes $\text{Av}(M_2, D^*)$ then $\bar{b}_0 \hat{\ } \bar{b}_\ell$ realizes $\text{tp}(\bar{a}_0 \hat{\ } \bar{a}_\ell, M_2)$ for $\ell = 1, 2$. So we are done. $\square_{1.5}$

1.6 Definition. Let $M \prec \mathfrak{C}, A \subseteq \mathfrak{C}$.

1) We define a universal first order theory $T_{M,A}$ as follows

- (a) the vocabulary is $\tau_{M,A} = \{P_{\varphi(\bar{x}, \bar{a})} : \varphi \in \mathbb{L}(\tau_T) \text{ and } \bar{a} \in {}^{\ell g(\bar{y})}A\} \cup \{c_a : a \in M\}$ with
 - (i) c_a an individual constant
 - (ii) $P_{\varphi(\bar{x}, \bar{a})}$ being a predicate with arity $\ell g(\bar{x})$; but we identify $P_{R(\bar{x})}$ with R so $\tau_T \subseteq \tau_{M,A}$
- (b) $T_{M,A}$ is the set of universal (first order) sentences satisfied in $\mathfrak{B}_{M,M,A}$, see part (2).

2) If $M \subseteq C \prec \mathfrak{C}$ and $\text{tp}(C, M \cup A)$ is f.s. in M (e.g., $C = M$) then we let $\mathfrak{B} = \mathfrak{B}_{C,M,A}$ be the $\tau_{M,A}$ -model with universe C such that $P_{\varphi(\bar{x}, \bar{a})}^{\mathfrak{B}} = \{\bar{b} \in {}^{\ell g(\bar{x})}C : \mathfrak{C} \models \varphi[\bar{b}, \bar{a}]\}$ for $\varphi(\bar{x}, \bar{y}) \in \mathbb{L}(\tau_T), \bar{a} \in {}^{\ell g(\bar{y})}(A)$ and such that $c_a^{\mathfrak{B}} = a$ for $a \in M$.

3) A model \mathfrak{B} of $T_{M,A}$

- (a) is called quasi-standard if $c_a^{\mathfrak{B}} = a$ for $a \in M$
- (b) is called standard if it is $\mathfrak{B}_{C,M,A}$ for some $C, M \subseteq C \subseteq \mathfrak{C}$.

4) Let $T_{M,A}^*$ be the model completion of $T_{M,A}$ (well defined only if it exists!)

1.7 Observation. 1) If $M \subseteq C, M \subseteq A$ and $\text{tp}(C, A)$ is f.s. in M , then $\mathfrak{B}_{C,M,A}$ is a model of $T_{M,A}$.

2) If \mathfrak{B} is a model of $T_{M,A}$, then \mathfrak{B} is isomorphic to the standard model $\mathfrak{B} = \mathfrak{B}_{C,M,A}$ of $T_{M,A,\mathfrak{B}}$ for some C .

3) Moreover, if $\mathfrak{B}_1 \subseteq \mathfrak{B}_2$ are models of $T_{M,A}$ and \mathfrak{B}_1 is standard, then \mathfrak{B}_2 is

isomorphic over \mathfrak{B}_1 to some standard \mathfrak{B}'_2 .

4) If $A_1 \subseteq A_2, M \subseteq C$, $\text{tp}(C, M \cup A_2)$ is f.s. in M then \mathfrak{B}_{C,M,A_1} is a reduct of \mathfrak{B}_{C,M,A_2} .

5) If $M \subseteq C_1 \subseteq C_2$ and $\text{tp}(C_2, M \cup A)$ is f.s. in M then $\mathfrak{B}_{C_1,M,A}$ is a submodel of $\mathfrak{B}_{C_2,M,A}$.

Proof. Easy.

1.8 Claim. Assume A is full over M .

1) $\mathfrak{B}_{M,M,A}$ is a model of $T_{M,A}$ with elimination of quantifier (in fact every definable set is definable by an atomic formula $R(x_0, \dots, x_{n-1})$).

2) If $\text{tp}(C, A)$ is f.s. in M then we can find M^+ such that

(a) $M \cup C \subseteq M^+ \prec \mathfrak{C}$

(b) $\text{tp}(M^+, A)$ is f.s. in M

(c) $\mathfrak{B}_{M^+,M,A}$ is an elementary extension of $\mathfrak{B}_{M,M,A}$.

3) $T_{M,A}$ has amalgamation and JEP.

4) $\text{Th}(\mathfrak{B}_{M,M,A})$ is the model completion of $T_{M,A}$ so is $T_{M,A}^*$.

5) $T_{M,A}^*$ is a dependent (complete first order) theory.

Proof. 1) By Claim 1.4(2), Definition 1.6(1) and A being full over M .

2) E.g. use an ultrapower \mathfrak{C}^κ/D of \mathfrak{C} with $\kappa \geq |T| + |C| + |A|$, D a regular filter on κ and let \mathbf{j} be the canonical embedding of \mathfrak{C} into \mathfrak{C}^κ/D . So we can find $f : C \rightarrow M^\kappa/D$ such that $f \cup (\mathbf{j} \upharpoonright A)$ is an elementary mapping, i.e., $(\mathfrak{C}, \mathfrak{C}^\kappa/D)$ -elementary embedding, now it should be clear.

3) By 1.5.

4) Check each case of $\mathfrak{B}_{M,M,A} \models (\forall \bar{y})(P_1(\bar{x}) \equiv \exists y P_2(y, \bar{x}))$ as above.

5) As $\mathfrak{B}_{M,M,A}$ is a model of it and reflects.

1.9 Conclusion. 1) $\text{Th}(\mathfrak{B}_{M,M,A})$ is a dependent (complete first order) theory.

2) $\kappa_{\text{ict}}(\text{Th}(\mathfrak{B}_{M,M,A})) = \kappa_{\text{ict}}(T)$, see §3.

Proof. 1) By 1.7(4) and 1.8(5) it is the duct of a dependent (complete first order) theory.

2) Similarly. □_{1.9}

1.10 Definition. 1) For any model \mathfrak{B} (not necessarily of T) and $A \subseteq \mathfrak{B}$ let $\mathbb{B}^m[A, \mathfrak{B}]$ be the family of subsets of ${}^m A$ of the form $\{\bar{a} \in {}^m A : \varphi(\bar{x}, \bar{a}) \in p\}$ for some $p \in \mathcal{S}^m(A, \mathfrak{B})$.
 2) If $\mathfrak{B} \prec \mathfrak{C}$ we may omit \mathfrak{B} .

Remark. If $\mathfrak{B} = \mathfrak{C}$ (or just if \mathfrak{B} is $|A|^+$ -saturated) then $\mathbb{B}^m[A, \mathfrak{B}] = \{\{\bar{a} : \mathfrak{B} \models \varphi[\bar{b}, \bar{a}]\} : \varphi(\bar{x}, \bar{y}) \in \mathbb{L}(\tau_{\mathfrak{B}}) \text{ and } \bar{b} \in {}^{\ell g(\bar{y})}\mathfrak{B}\}$.

1.11 Question: Assume $M \subseteq A$ and \mathfrak{B} a standard model of $T_{M,A}$ and $N = \mathfrak{B} \upharpoonright \tau_T$. Then do we have

$(*)_{T, T_{M,A}}$ for any ultrafilter D_0 on $\mathbb{B}[N]$, the number of ultrafilters D_1 on $\mathbb{B}[\mathfrak{B}, \mathfrak{B}]$ extending it is at most $2^{|T|+|A|}$.

1.12 Remark. 1) For complete (first order theories) $T \subseteq T_1$, the condition $(*)_{T, T_1}$ of 1.11 has affinity to conditions like “any model of T has $\leq 1/ < \aleph_0/ \leq \|M\|$ expansions to a model of T_1 ”. What is the syntactical characterization?

2) When is $\mathfrak{B}_{N,M,A}$ a model of $T_{M,A}^*$?

$\Box_{N,M,A}$ every formula over $N \cup A$ which does not fork over N is realized in N .

§2 MORE ON INDISCERNIBLE SEQUENCES

This section is complimentary to [Sh 715, §5].

2.1 Claim. *Assume*

- (α) $\bar{\mathbf{b}} = \langle \bar{b}_t : t \in I_0 \rangle$ is an infinite indiscernible sequence over A
- (β) $B \subseteq \mathfrak{C}$.

Then we can find I_1, J and \bar{b}_t for $t \in I_1 \setminus I_0$ such that:

- (a) $I_0 \subseteq I_1, I_1 \setminus I_0 \subseteq J \subseteq I_1$ and $|I_1 \setminus I_0| \leq |J| \leq |B| + |T|$
- (b) $\bar{\mathbf{b}}' = \langle \bar{b}_t : t \in I_1 \rangle$ is an indiscernible sequence over A
- (c) if I_2 is a J -free extension of I_1 (see below) and \bar{b}_t for $t \in I_2 \setminus I_1$ are such that $\bar{\mathbf{b}}'' = \langle \bar{b}_t : t \in I_2 \rangle$ is an indiscernible sequence over A then
 - ⊗ if $n < \omega, \bar{s}, \bar{t} \in {}^n(I_2)$ and $\bar{s} \sim_J \bar{t}$ (recall that this means that $(s_\ell <_{I_2} s_k) \equiv (t_\ell <_{I_2} t_k)$ and $(s_\ell <_{I_2} r) \equiv (t_\ell <_{I_2} r), (r <_{I_2} s_\ell \equiv r <_{I_2} t_\ell)$ whenever $\ell, k < n, r \in I_1$), then $\bar{b}_{\bar{s}}, \bar{b}_{\bar{t}}$ realizes the same type over $A \cup B$ where $\bar{b}_{\langle t_\ell : \ell < n \rangle} = \bar{b}_{t_0} \wedge \bar{b}_{t_1} \wedge \dots \wedge \bar{b}_{t_{n-1}}$.

2.2 Definition. For linear orders $J \subseteq I_1 \subseteq I_2$ we say that I_2 is a J -free extension of I_1 when: ($J \subseteq I_1 \subseteq I_2$ and)

- ⊗ if $t \in I_2 \setminus I_1$ then for some $s_1, s_2 \in I_2$ we have $s_1 <_{I_2} t <_{I_2} s_2$ and $[s_1, s_2]_{I_1} \cap J = \emptyset$.

2.3 Remark. Why do we need “ J -free”? Let $M = (\mathbb{R}, <, Q^M), Q^M = \mathbb{Q}, B = \{0\}, A = \emptyset, I_0$ the irrationals, $b_t = t$ for $t \in I_0$.

Proof. We try to choose by induction on $\zeta < \lambda^+$ where $\lambda = |T| + |B|$ a sequence $\bar{\mathbf{b}}^\zeta = \langle \bar{b}_t : t \in J_\zeta \rangle$ and $n_\zeta, \bar{s}_\zeta, \bar{t}_\zeta, J'_\zeta, \varphi_\zeta, \bar{c}_\zeta, \bar{d}_\zeta$ such that

- (a) J_ζ is a linear order, increasing continuous with ζ
- (b) $J_0 = I_0$ (so $\bar{\mathbf{b}}^0 = \bar{\mathbf{b}}$), $J_{\varepsilon+1} \setminus J_\varepsilon$ is finite so $|J_\varepsilon \setminus I_0| < |\varepsilon|^+ + \aleph_0$
- (c) $\bar{\mathbf{b}}^\zeta$ is an indiscernible sequence over A
- (d) $J'_\zeta \subseteq J_\zeta, J_\zeta = I_0 \cup J'_\zeta, J'_\zeta$ is increasing continuous in ζ and $|J'_\zeta| < |\zeta|^+ + \aleph_0$

- (e) if $\zeta = \varepsilon + 1$ then $n_\varepsilon < \omega$, $\bar{s}_\varepsilon \in {}^{n_\varepsilon}(J'_\zeta)$, $\bar{t}_\varepsilon \in {}^{n_\varepsilon}(J'_\zeta)$, $\varphi_\varepsilon = \varphi_\varepsilon(\bar{x}_0, \dots, \bar{x}_{n_\varepsilon-1}, \bar{c}_\varepsilon, \bar{d}_\varepsilon)$, $\bar{c}_\varepsilon \subseteq B$, $\bar{d}_\varepsilon \subseteq A$ and $J'_\zeta = J'_\varepsilon \cup (\bar{s}_\zeta \hat{\ } \bar{t}_\zeta)$
- (f) $\bar{s}_\varepsilon \sim_{J'_\varepsilon} \bar{t}_\varepsilon$ and $\models \varphi_\varepsilon[\bar{b}_{\bar{s}_\varepsilon}, \bar{c}_\varepsilon, \bar{d}_\varepsilon]$ & $\neg \varphi_\varepsilon[\bar{b}_{\bar{t}_\varepsilon}, \bar{c}_\varepsilon, \bar{d}_\varepsilon]$
- (g) $J_{\zeta+1}$ is a J'_ζ -free extension of J_ζ .

If we succeed, for some unbounded $w \subseteq \lambda^+$ we have $\varepsilon \in w \Rightarrow n_\varepsilon = n_*$, $\varphi_\varepsilon = \varphi_*$, $\bar{c}_\varepsilon = \bar{c}^*$, $\bar{d}_\varepsilon = \bar{d}^*$; then we let $J^* = \cup\{J'_\zeta : \zeta < \lambda^+\}$, so every $J' \subseteq J^*$ of cardinality $\leq \lambda$ is included in J'_ζ for some $\zeta < \lambda^+$ and we get contradiction to [Sh 715, 3.2](=3.4t), hence we fail. For $\zeta = 0$ all is O.K., and for ζ limit there is no problem. So we are stuck at stage $\zeta = \varepsilon + 1$, i.e. \mathbf{b}^ε is defined but we cannot choose $\bar{\mathbf{b}}^\zeta$. Then $\bar{\mathbf{b}}^\varepsilon$ is as required.

□_{2.1}

The aim of 2.4 + 2.7 below is to show a complement of [Sh 715, §5]; that is, in the case of small cofinality what occurs in one cut is the “same” as what occurs in others.

2.4 Claim. *Assume*

- (a) $\mu \geq |T|$
- (b) I_ℓ for $\ell < 4$ are pairwise disjoint linear orders
- (c) $I_\ell = \bigcup_{\beta < \mu^+} I_\ell^\beta$, I_ℓ^β (strictly) increasing with β and $|I_\ell^\beta| \leq \mu$ for $\ell < 4$
- (d) (α) $\ell \in \{0, 2\} \Rightarrow I_\ell^\beta$ an end segment of I_ℓ
 (β) $\ell \in \{1, 3\} \Rightarrow I_\ell^\beta$ is an initial segment of I_ℓ
- (e) $I = I_0 + I_1 + I_2 + I_3$ and $I^\beta = I_0^\beta + I_1^\beta + I_2^\beta + I_3^\beta$
- (f) $\langle \bar{b}_t : t \in I \rangle$ is an indiscernible sequence.

Then we can find a limit ordinal $\beta(*) < \mu^+$ and $\langle \bar{b}_t^* : t \in I \rangle$ such that:

- (A) $\bar{b}_t^* = \bar{b}_t$ if $t \in I \setminus I^{\beta(*)}$
- (B)₁ $\langle \bar{b}_t^* : t \in I \setminus I_0^{\beta(*)} \setminus I_1^{\beta(*)} \rangle$ is an indiscernible sequence
- (B)₂ $\langle \bar{b}_t^* : t \in I \setminus I_2^{\beta(*)} \setminus I_3^{\beta(*)} \rangle$ is an indiscernible sequence
- (C)₁ $\text{tp}_*(\langle \bar{b}_t^* : t \in I_0^{\beta(*)} \cup I_1^{\beta(*)} \rangle, \cup\{\bar{b}_t^* : t \in (I \setminus I^\beta) \cup I_2^{\beta(*)} \cup I_3^{\beta(*)} \cup (I_0^{\beta(*)} + \omega \setminus I_0^{\beta(*)}) \cup (I_1^{\beta(*)} + \omega \setminus I_1^{\beta(*)})\}) \vdash$
 $\text{tp}_*(\langle \bar{b}_t^* : t \in I_0^{\beta(*)} \cup I_1^{\beta(*)} \rangle, \cup\{\bar{b}_t^* : t \in (I \setminus I^{\beta(*)}) \cup I_2^{\beta(*)} \cup I_3^{\beta(*)}\})$ for any $\beta \in [\beta(*), \mu^+)$

- (C)₂ $\text{tp}_*(\langle \bar{b}_t^* : t \in I_2^{\beta(*)} \cup I_3^{\beta(*)} \rangle, \cup \{ \bar{b}_t^* : t \in (I \setminus I^\beta) \cup I_0^{\beta(*)} \cup I_1^{\beta(*)} \cup (I_2^{\beta(*)} + \omega \setminus I_2^{\beta(*)}) \cup (I_3^{\beta(*)} + \omega \setminus I_3^{\beta(*)}) \}) \vdash$
 $\text{tp}(\langle b_t^* : t \in I_2^{\beta(*)} \cup I_3^{\beta(*)} \rangle, \cup \{ \bar{b}_t^* : t \in (I \setminus I^{\beta(*)}) \cup I_0^{\beta(*)} \cup I_1^{\beta(*)} \})$ for any $\beta \in [\beta(*), \mu^+)$
- (D)₁ $\langle \bar{b}_t^* : t \in I_0 \setminus I_0^{\beta(*)} \rangle$ is an indiscernible sequence over $\cup \{ \bar{b}_t^* : t \in I_0^{\beta(*)} \cup I_1 \cup I_2 \cup I_3 \}$
- (D)₂ $\langle \bar{b}_t^* : t \in (I_1 \setminus I_1^{\beta(*)}) + (I_2 \setminus I_2^{\beta(*)}) \rangle$ is an indiscernible sequence over $\cup \{ \bar{b}_t^* : t \in I_0 \cup I_1^{\beta(*)} \cup I_2^{\beta(*)} \cup I_3 \}$
- (D)₃ $\langle \bar{b}_t^* : t \in I_3 \setminus I_3^{\beta(*)} \rangle$ is an indiscernible sequence over $\cup \{ \bar{b}_t^* : t \in I_0 + I_1 + I_2 + I_3^{\beta(*)} \}$.

2.5 Remark. What occurs if T is stable (or just $\bar{\mathbf{b}}$ is)? we get something like $\{ \bar{b}_t^* : t \in I_0^{\beta(*)} \cup I_1^{\beta(*)} \} = \{ \bar{b}_t^* : t \in I_2^{\beta(*)} \cup I_3^{\beta(*)} \}$.

Proof. For simplicity assume $I_\ell^0 \neq \emptyset$.

We choose by induction on $n < \omega$ an ordinal $\beta(n)$ and $\langle b_t^n : t \in I \rangle$ such that:

- (α) $\beta(n) < \mu^+, \beta(0) = 0, \beta(n) < \beta(n+1)$
- (β) $\bar{b}_t^n = \bar{b}_t$ if $t \in I \setminus I^{\beta(n)}$ or if $n = 0$
- (γ)₁ $\langle \bar{b}_t^n : t \in I \setminus I_0^{\beta(n)} \setminus I_1^{\beta(n)} \rangle$ realizes the same type as $\langle \bar{b}_t : t \in I \setminus I_0^{\beta(n)} \setminus I_1^{\beta(n)} \rangle$
- (γ)₂ $\langle \bar{b}_t^n : t \in I \setminus I_2^{\beta(n)} \setminus I_3^{\beta(n)} \rangle$ realizes the same type as $\langle \bar{b}_t : t \in I \setminus I_2^{\beta(n)} \setminus I_3^{\beta(n)} \rangle$
- (δ)₁ if n is even then:
 - (1) $\bar{b}_t^{n+1} = \bar{b}_t^n$ for $t \in I \setminus I_2^{\beta(n)} \setminus I_3^{\beta(n)}$
 - (2) if $\beta(n+1) < \beta < \mu^+$ then the type which $\langle \bar{b}_t^{n+1} : t \in I_2^{\beta(n)} + I_3^{\beta(n)} \rangle$ realizes over $\cup \{ \bar{b}_t^n : t \in (I_0 \setminus I_0^\beta) \cup I_0^{\beta(n+1)} \cup (I_1 \setminus I_1^\beta) \cup I_1^{\beta(n+1)} \cup (I_2 \setminus I_2^\beta) \cup (I_2^{\beta(n+1)} \setminus I_2^{\beta(n)}) \cup (I_3 \setminus I_3^\beta) \cup (I_3^{\beta(n+1)} \setminus I_3^{\beta(n)}) \}$ has a unique extension over $\cup \{ b_t^n : t \in I \setminus I_2^{\beta(n)} \cup I_3^{\beta(n)} \}$
 - (3) $\bar{b}_t^{n+1} = b_t^n$ if $t \in I_2^{\beta(k)} \cup I_3^{\beta(k)}, k < n$
- (δ)₂ if n is odd like (δ)₁ inverting the roles of $(I_0, I_1), (I_2, I_3)$
- (ε) $\langle \bar{b}_t^n : t \in I \rangle$ satisfies clauses (D)₁, (D)₂, (D)₃ of the claim with $\beta(n)$ instead of $\beta(*)$.

The induction step is as in the proof of 2.1 (though we use the finite character for the middle clause (2) of clauses $(\delta)_1, (\delta)_2$). Alternatively, define $\beta(\varepsilon), \langle \bar{b}_t^\varepsilon : t \in I \rangle$ for $\varepsilon < \mu^+$ each time just having in $(\delta)_\ell$ an extension witnessing the failure of the desired conclusion.

Alternatively, letting n be even we try to choose $\beta_n(\varepsilon), \bar{\mathbf{b}}^{n,\varepsilon}, \langle \bar{b}_t^\varepsilon : t \in I_2^{\beta(n)} + I_3^{\beta(n)} \rangle$ by induction on $\varepsilon \leq \mu^+$ such that:

- $\odot(a)$ $\beta_n(\varepsilon) < \mu^+$
- (b) $\beta_n(0) = \beta(n)$
- (c) $\beta_n(\varepsilon)$ is increasing continuous
- (d) $\zeta < \varepsilon \Rightarrow \text{tp}(\bar{\mathbf{b}}^{n,\varepsilon} \cup \{b_t^n : t \in (I \setminus I_{\beta_n(\varepsilon)}^\varepsilon) \cup I_{\beta_n(\zeta)}\}),$
 $\text{tp}(\bar{\mathbf{b}}^{n,\zeta} \cup \{\bar{b}_t^n : t \in (I \setminus I_{\beta_n(\varepsilon)}^\varepsilon) \cup I_{\beta_n(\zeta)}\})$
- (e) if $\varepsilon = \zeta + 1$, this witness then $(\delta)_1(2)$ fails if we let

$$\bar{b}_t^{n+1} = \begin{cases} b_t^n & \text{if } t \in I \setminus I_2^{\beta(n)} \setminus I_3^{\beta(n)} \\ \bar{b}_t^{n,\zeta} & \text{if } t \in I_2^{\beta(n)} \cup I_3^{\beta(n)} \end{cases}.$$

If we succeed to carry the induction, by [Sh 715] for some ε , the sequence $\langle \bar{b}_t^n : t \in I_0^{\beta_n(\varepsilon)} \rangle, \langle \bar{b}_t^n : t \in I_1^{\beta_n(\varepsilon)} + I_2^{\beta_n(\varepsilon)} \rangle, \langle \bar{b}_t^n : t \in I_3^{\beta_n(\varepsilon)} \rangle$ are mutually indiscernible over $\cup \{\bar{b}_t^{n,\mu^+} : t \in I_2^{\beta(n)} + I_3^{\beta(n)}\} \cup \{b_t^n : t \in (I \setminus I_{\beta_n(\varepsilon)}^\varepsilon)\}$ (because $\langle \bar{b}_t : t \in I_0 \setminus I_0^{\beta_n(\varepsilon)} \rangle, \langle \bar{b}_t : t \in (I_1 \setminus I_{\beta_n(\varepsilon)}^\varepsilon) + I_2 \setminus I_2^{\beta_n(\varepsilon)} \rangle, \langle \bar{b}_t : t \in I_3 \setminus I_3^{\beta_n(\varepsilon)} \rangle$ are mutually indiscernible, recalling (β)).

This contradicts (e) . So we cannot complete the induction. We certainly succeed for $\varepsilon = 0$, and there is no problem for limit $\varepsilon \leq \mu^+$. So for some $\varepsilon = \zeta + 1$ we have succeed for ζ and cannot choose for ε . We define \bar{b}_i^{n+1} as in (e) of \odot above, and choose $\beta(n+1) \in [\beta_n(\varepsilon), \mu^+)$ such that clauses (ε) holds.

Let $\beta(*) = \cup \{\beta(n) : n < \omega\} < \mu^+$, \bar{b}_t^* is \bar{b}_t^n for every n large enough (exists by clause (β) if $t \in I \setminus I^{\beta(*)}$ and by $(\delta)_\ell(i) + (iii)$ if $t \in I^{\beta(*)}$).

Clearly we are done. $\square_{2.4}$

2.6 Claim. Assume

- (a) $I, I^\beta, I_\ell, I_\ell^\beta$ for $\ell < 4, \beta < \mu^+$ are as in the assumption of claim 2.4
- (b) $\beta(*)$ and $\langle \bar{b}_t^* : t \in I \rangle$ and as in the conclusion of claim 2.6
- (c) $J^+ = J_0^+ + J_1^+ + J_2^+ + J_3^+ + J_4^+$ linear orders
- (d) $J = J_0 + J_1 + J_2 + J_3 + J_4$ linear orders
- (e) $J_1 = J_1^+ + I_0^{\beta(*)} + I_1^{\beta(*)}$ and $J_3 = I_2^{\beta(*)} + I_3^{\beta(*)}$
- (f) $J_0 \subseteq J_0^+$ and $I_0 \setminus I_0^{\beta(*)} \subseteq J_0^+$

- (g) $J_2 \subseteq J_2^+$ and $(I_1 \setminus I_1^{\beta(*)}) + (I_2 \setminus I_2^{\beta(*)}) \subseteq J_2$
- (h) $J_4 \subseteq J_4^+$ and $(I_3 \setminus I_3^{\beta(*)}) \subseteq J_4^+$
- (i) $\langle \bar{b}_t^* : t \in J^+ \rangle$ is an indiscernible sequence.

1) If J'_0, J'_2, J'_4 are infinite initial segments of J_0, J_2, J_4 respectively and J''_0, J''_2, J''_4 are infinite end segments of J_0, J_2, J_4 respectively.

Then

- (α) $\text{tp}(\langle \bar{b}_t^* : t \in J_3 \rangle \cup \{\bar{b}_s : s \in J'_0 \cup J_1 \cup J'_2 \cup J'_4\}) \vdash \text{tp}(\langle \bar{b}_t^* : t \in J_3 \rangle, \cup \{\bar{b}_s^* : s \in J_0 \cup J_1 \cup J_2 \cup J_4\})$
- (β) like (α) interchanging J_3, J_1 .

2) If J_0 has no first element, $J'_0 \subseteq J_0$ is unbounded from below, $J'_2 \subseteq J_2$ is infinite and J_4 has no last element and $J'_4 \subseteq J_4$ is unbounded from above, then the conclusions of (1) holds

- (α) $\text{tp}(\langle \bar{b}_t^* : t \in J_3 \rangle, \cup \{\bar{b}_s : s \in J'_0 \cup J''_0 \cup J_1 \cup J'_2 \cup J''_2 \cup J'_4 \cup J''_4\}) \vdash \text{tp}(\langle \bar{b}_t^* : t \in J_3 \rangle, \cup \{\bar{b}_s^* : s \in J_0 \cup J_1 \cup J_2 \cup J_4\})$
- (β) $\text{tp}(\langle \bar{b}_t^* : t \in J_1 \rangle, \cup \{\bar{b}_s : s \in J'_0 \cup J''_0 \cup J'_2 \cup J''_2 \cup J_3 \cup J'_4 \cup J''_4\}) \vdash \text{tp}(\langle \bar{b}_t^* : t \in J_1 \rangle, \cup \{\bar{b}_s^* : s \in J_0 \cup J_2 \cup J_3 \cup J_4\})$.

3) If J_0^*, J_2^*, J_4^* has neither first element nor last element and J'_0, J'_2, J'_4 are subsets of J_0, J_2, J_4 respectively unbounded from below and J''_0, J''_2, J''_4 are subsets of J_0, J_2, J_4 respectively unbounded from above, then the conclusion of part (1) holds.

Proof. By the local character of \vdash and by the indiscernibility demands in 2.4, i.e., $(D)_1, (D)_2, (D)_3$. $\square_{2.6}$

2.7 Conclusion. 1) If $\mu \geq \kappa \geq |T|$, then for some linear order J^* of cardinality κ we have

- $\boxtimes_{\bar{b}^*, J^*}$ if
 - (a) $J = J_0 + J_1 + J_2 + J_3 + J_4$
 - (b) the cofinalities of J_0, J_2, J_4 and there inverse are $\leq \mu$ but are infinite
 - (c) $J_1 \cong J^*$ and $J_3 \cong J^*$ (hence J_1, J_3 has cardinality $\leq \kappa$)
 - (d) $\langle \bar{b}_t : t \in J \setminus J_3 \rangle$ is an indiscernible sequence (of m -tuples)
 - (e) M is a μ^+ -saturated model
 - (f) $\cup \{\bar{b}_t : t \in J \setminus J_3\} \subseteq M$.

Then we can find $\bar{b}_t \in {}^m M$ for $t \in J_3$ such that $\langle \bar{b}_t : t \in I \setminus J_2 \rangle$ is an indiscernible sequence.

2) If we allow J^* to depend on $\text{tp}'(\bar{\mathbf{b}}^*)$, then we can use J^* of the form $\delta^* + \delta$, $\delta < \mu^+$ (δ^* - the inverse of δ).

2.8 Definition. For an infinite indiscernible sequence $\langle \bar{b}_t : t \in J \rangle$ over A let $\text{tp}'(\bar{\mathbf{b}}, A) = \langle p_n : n < \omega \rangle$ when $p_n = \text{tp}(\bar{b}_{t_0} \wedge \dots \wedge \bar{b}_{t_{n-1}}, A)$ for any $t_0 <_J t_1 <_J \dots <_J t_{n-1}$.

Proof. Let $\bar{\mathbf{b}}^\otimes$ be an infinite indiscernible sequence.

Let J_0, J_2, J_4 be disjoint linear orders as in (b), Apply 2.4 with I_1, I_3 isomorphic to $(\mu^+, <)$ and I_0, I_2 isomorphic to $(\mu^+, >)$, say $I_\ell = \{t_\alpha^\ell : \alpha < \mu^+\}$ with t_α^ℓ increasing with α if $\ell \in \{1, 3\}$ and decreasing with α if $\ell \in \{0, 2\}$, we get $\bar{b}^* = \langle b_t^* : t \in \sum_{\ell < 4} I_\ell \rangle, \beta(*)$ as there with $\text{tp}'(\bar{\mathbf{b}}^* \upharpoonright I_0) = \text{tp}'(\bar{\mathbf{b}}^\otimes)$,

see Definition 0.1. Let $J_0^+ = J_0 + (I_0 \setminus I_0^{\beta(*)}), J_1^+ = J_1 = I_0^{\beta(*)} + I_1^{\beta(*)}, J_2^+ = J_2 + (I_1 \setminus I_1^{\beta(*)}) + (I_2 \setminus I_2^{\beta(*)}), J_3 = I_2^{\beta(*)} + I_3^{\beta(*)} + J_3$ and $J_4^+ = J_4 + (I_3 \setminus I_3^{\beta(*)})$ and $J^+ = J_0^+ + J_1^+ + J_2^+ + J_3 + J_4^+$. All J_ℓ are infinite linear orders, choose $J^* = J_1$, clearly $J_3 \cong J^*$. Now

(*) $\langle \bar{b}_t^* : t \in J \setminus J_5 \rangle$ is an indiscernible sequence and

(**) if $M \supseteq \cup \{\bar{b}_t^* : t \in J \setminus J_3\}$ is μ^+ -saturated then we can find $\bar{b}'_t \in {}^m M$ for $t \in J_3$ such that

$$\langle \bar{b}_t^* : t \in J_0 \rangle \wedge \langle \bar{b}'_t : t \in J_3 \rangle \wedge \langle \bar{b}_t^* : t \in J_4 \rangle$$

is an indiscernible sequence.

[Why? Choose $J'_0 \subseteq J_0$ unbounded from below of cardinality $\text{cf}(J_0, >_{J_0})$ which is $\leq \mu$ but $\geq \aleph_0$ and similarly $J'_2 \subseteq J_2, J'_4 \subseteq J_4$ and choose $J''_0 \subseteq J_0$ unbounded from above of cardinality $\text{cf}(J_0)$ which is $\leq \mu$ and similarly $J''_2 \subseteq J_2, J''_4 \subseteq J_4$ (all O.K. by clause (b) of the assumption).

Now $p = \text{tp}(\langle \bar{b}_t^* : t \in J_3 \rangle, \cup \{\bar{b}_s : s \in J'_0 \cup J''_0 \cup J'_2 \cup J''_2 \cup J'_4 \cup J''_4\})$ is a type of cardinality $\leq |T| + |J'_0| + |J''_0| + |J'_2| + |J''_2| + |J'_4| + |J''_4| \leq \mu$ hence is realized by some sequence $\langle \bar{b}'_t : t \in J_3 \rangle$ from M .

By Claim 2.6 the desired conclusion in (**) holds.]

So we have gotten the desired conclusion for any $\langle J_\ell : \ell \leq 4 \rangle$ and indiscernible sequence, $\bar{\mathbf{b}} = \langle \bar{b}_t : t \in J \setminus J_5 \rangle$ as long as $\text{tp}'(\bar{\mathbf{b}}) = \text{tp}'(\bar{\mathbf{b}}^\otimes)$ and the order type of J_1, J_3 is as required for $\bar{\mathbf{b}}^\otimes$. This is enough for part (2), we are left with (1).

Note that by the proof of 2.1, the set of $\beta(*)$ as required contains $E \cap \{\delta < \mu^+ : \text{cf}(\delta) = \aleph_0\}$ for some club E (in fact even contains E). So if $\mu \geq 2^{|T|}$, as $\{\text{tp}'(\bar{\mathbf{b}}) : \bar{\mathbf{b}} \text{ an infinite indiscernible sequence}\}$ has cardinality $\leq 2^{|T|}$ we are done.

Otherwise choose J^* a linear order of cardinality μ isomorphic to its inverse, to $J^* \times \omega$ and to $J^* \times (\gamma + 1)$ ordered lexicographically for every $\gamma \leq \mu$ (hence for every $\gamma < \mu^+$), (e.g. note if J^{**} is dense with no first and last element and saturated (or special) of cardinality $> \mu$, then $J^{**} \times \omega$ satisfies this and use the L.S. argument). So we can in 2.4 hence 2.6 use I_ℓ ($\ell < 4$) such that $I_\ell^{\beta+1} \cong J^*$ for $\beta < \mu^+$, $\ell < 4$. So $I_0^{\beta(*)} + I_1^{\beta(*)} \cong J^* \cong I_2^{\beta(*)} + I_3^{\beta(*)}$. $\square_{2.7}$

2.9 Conclusion. In 2.7:

- (A) we can choose $J^* = \mu^* + \mu$ i.e. $\{0\} \times (\mu, >) + \{1\} \times (\mu, <)$
- (B) if J is a linear order ($\neq \emptyset$) or cardinality $\leq \mu$, we can have $J^* = (\mu^* + \mu) \times J$ ordered lexicographically
- (C) we can change the conclusion of 2.7 to make it symmetrical between J_3 and J_5
- (D) we use only clause $(E)_2$ of 2.4, we could use clause $(E)_1$, too.

Proof. (A),(B) combine the proofs of 2.1 and 2.4 trying to contradict each formula, by bookkeeping trying for it enough times. $\square_{2.9}$

We may look at it differently, part (2) is closed in formulation to be a complement to §5.

2.10 Conclusion. 1) Assume

- (a) $J = I \times J^*$ lexicographically, J^*, μ is as in 2.7, I infinite
- (b) $\langle \bar{b}_t : t \in J \rangle$ an indiscernible sequence, $\ell g(\bar{b}_t) = m$ or just $\ell g(\bar{b}_t) < \mu^+$
- (c) for $s \in I$ let \bar{c}_s be $\langle \bar{b}_t : t \in \{s\} \times J^* \rangle$ more exactly the concatenation.

Then

- (α) $\langle \bar{c}_s : s \in I \rangle$ is an infinite indiscernible sequence
- (β) is $s_0 <_I \dots <_I s_7$ then there is \bar{c} realizing $\text{tp}(\bar{c}_{s_2}, \cup \{\bar{c}_{s_\ell} : \ell \leq 7, \ell \neq 2\})$ such that $\text{tp}(\bar{c}', \cup \{\bar{c}_{s_\ell} : \ell \leq 7, \ell \neq 2\}) + \text{tp}(\bar{c}_{s_2}, \cup \{\bar{c}_s : s_0 \leq_I s \leq_I s_1 \text{ or } s_3 \leq_I s \leq_I s_4 \text{ or } s_6 \leq_I s \leq_I s_7\})$
- (γ) similarly inverting the order (i.e. interchanging the roles of s_2, s_5 in clause (β)).

2) The sequence $\langle \bar{c}_s : s \in I \rangle$ from part (1) satisfies $M \supseteq \cup \{\bar{c}_s : s \in I\}$ and $(I_1, I_2), (I_3, I_4)$ are a Dedekind cut of I , each of $I_1, (I_2)^*, I_3(I_4)^*$ is non empty of cofinality $\leq \mu$ and let $I^+ \supseteq I, t_2, t_5 \in I_1^+$ realizes the cuts $(I_1, I_2), (I_3, I_4)$ respectively and \bar{c}_t for $t \in I^+ \setminus I$ are such that $\langle \bar{c}_t : t \in I^+ \rangle$ is indiscernible (then for notational simplicity), then

- there is a sequence in M realizing $\text{tp}(\bar{c}_{t_2}), \cup \{\bar{c}_s : s \in I\}$ iff there is a sequence in M realizing $\text{tp}(\bar{c}_{t_5}), \cup \{\bar{c}_s : s \in I\}$.

Concluding Remark. There is a gap between [Sh 715, 5.11=np5.5] and the results in §2, some light is thrown by

2.11 Claim. In [Sh 715, 5.11=np5.5]; we can omit the demand $\text{cf}(\text{Dom}(\bar{\mathbf{a}}^\zeta)) \geq \kappa_1$ (= clause (f) there) if we add $\zeta < \zeta^* \Rightarrow (\theta_\zeta^1)^+ = \lambda$.

Proof. By the omitting type argument.

2.12 Question: Assume:

- (a) $\langle (N_i, M_i) : i \leq \kappa \rangle$ is \prec -increasing (as pairs), M_{i+1}, N_{i+1} are λ_i^+ -saturated, $\|N_i\| \leq \lambda_i, \langle \lambda_i : i < \kappa \rangle$ increasing, $\kappa < \lambda_0$
- (b) $p(\bar{x})$ is a partial type over $N_0 \cup M_\kappa$ of cardinality $\leq \lambda_0$.

- 1) Does $p(\bar{x})$ have a λ_0^+ -isolated extension?
- 2) Does this help to clarify DOP?
- 3) Does this help to clarify “if any M is a benign set” (see [BBSH 815]).

2.13 Claim. Assume

- (a) M is λ^+ -saturated
- (b) $p(\bar{x})$ is a type of cardinality $\leq \kappa, \ell g(\bar{x}) \leq \kappa$
- (c) $\text{Dom}(p) \subseteq A \cup M, |A| \leq \kappa \leq \lambda$
- (d) $B \subseteq M, |B| \leq \lambda$.

Then there is a type $q(\bar{x})$ over $A \cup M$ of cardinality $\leq \kappa$ and $r(\bar{x}) \in S^{\ell g(\bar{x})}(A \cup B)$ such that

$$p(\bar{x}) \subseteq q(\bar{x})$$

$$q(\bar{x}) \vdash r(\bar{x})$$

Remark. This defines a natural quasi order (type definable) is it directed?

§3 STRONGLY DEPENDENT THEORIES

3.1 *Context.* T complete first order, \mathfrak{C} a monster model.

3.2 Definition. 1) T is strongly dependent if:

there are no $\bar{\varphi} = \langle \varphi_n(\bar{x}, \bar{y}_n) : n < \omega \rangle$ and $\langle \bar{a}_\alpha^n : n < \omega, \alpha < \lambda \rangle$ such that

- (a) for every $\eta \in {}^\omega \lambda$ the set $p_\eta = \{\varphi_n(\bar{x}, \bar{a}_\alpha^n) : n < \omega\}$ is consistent; so $\ell g(\bar{a}_\alpha^n) = \ell g(\bar{y}_n)$
- (b) for each $n < \omega$ for some k_n any k_n of the formulas $\{\varphi_n(\bar{x}, \bar{a}_\alpha^n) : \alpha < \lambda\}$ are contradictory.

2) T is strongly stable if it is stable and strongly dependent.

3) $\kappa_{i < t}(T)$ is the first κ such that there is no $\bar{\varphi} = \langle \varphi_\alpha(\bar{x}, \bar{y}_\alpha) : \alpha < \kappa \rangle$ satisfying the parallel of part (1).

3.3 Claim. 1) If T is superstable, then T is strongly dependent.

2) If T is strongly dependent, then T is dependent.

3) There are stable T which are not strongly dependent.

4) There are stable not superstable T which are strongly dependent.

5) There are unstable strongly dependent theories.

6) The theory of real closed fields is strongly dependent.

7) If T is stable then $\kappa_{\text{ict}}(T) \leq \kappa(T)$.

Proof. Easy. E.g.

3) E.g. $T = \text{Th}({}^\omega \omega, E_n^1)_{n < \omega}$ where $\eta E_n^1 \nu \Leftrightarrow \eta(n) = \nu(n)$.

4) E.g., $T = \text{Th}({}^\omega \omega, E_n^2)_{n < \omega}$ where $(\eta E_n^2 \nu) \equiv (\eta \upharpoonright n = \nu \upharpoonright n)$.

5) E.g., $T = \text{Th}(\mathbb{Q}, <)$, the theory of dense linear orders with no first and no last element. □_{3.3}

3.4 Definition. 1) We say a type $p(x)$ is a $(1 = \aleph_0)$ -type (or p satisfies $1 = \aleph_0$) if for some set A for every countable set $B \subseteq p(\mathfrak{C})$, for some $a \in p(\mathfrak{C})$ we have $B \subseteq \text{acl}(\{a\} \cup A)$.

1A) If $A = \text{Dom}(p)$ we add purely. We call A a witness to $p(x)$ being a $(1 = \aleph_0)$ -type.

2) We say T is a local or global $(1 = \aleph_0)$ -theory if for some A (the witness) some or every non algebraic type $p \in \mathcal{S}(A)$ is a $(1 = \aleph_0)$ -type. If $A = \emptyset$ we say purely.

3) We say that a type $p(x)$ is a weakly $(1 = \aleph_0)$ -type if: for some set $A \subseteq p(\mathfrak{C})$ for every indiscernible sequence $\langle a_n : n < \omega \rangle$ over A such that $a_n \in p(\mathfrak{C})$ there

is $a \in p(\mathfrak{C})$ such that $\{a_n : n < \omega\} \subseteq \text{acl}\{a\} \cup (A)$. We let “purely”, “witness” “local”; “global” be defined similarly.

- 3.5 Observation.* 1) Every algebraic type $p(x)$ is a $(1 = \aleph_0)$ -type.
 2) If $p(x)$ is a $(1 = \aleph_0)$ -type, then it has a witness A , $|A| \leq 2^{|\text{Dom}(p)+|T|}$.
 3) If $p(x)$ is a $(1 = \aleph_0)$ -type then $p(x)$ is a weakly $(1 = \aleph_0)$ -type.

Proof. Easy.

3.6 Claim. *If T is strongly dependent, then no non algebraic type is $(1 = \aleph_0)$ -type even in \mathfrak{C}^{eq} and no non-algebraic type is a weakly $(1 = \aleph_0)$ -type.*

Proof. Let $\lambda \geq \aleph_0$. Assume toward contradiction that $p(x)$ is a non-algebraic type and A a witness for it. As $p(x)$ is not algebraic, we can find $\bar{b}^n = \langle b_\alpha^n : \alpha < \lambda \rangle$ for $n < \omega$ such that

- (*)₁ b_ℓ^n realizes p
- (*)₂ $b_\alpha^n \neq b_\beta^n$ for $\alpha < \beta < \lambda, n < \omega$
- (*)₃ $\langle b_\alpha^n : (n, \alpha) \in \omega \times \lambda \rangle$ is an indiscernible sequence over A where $\omega \times \lambda$ is ordered lexicographically.

Let $a \in p(\mathfrak{C})$ be such that $\{b_0^n : n < \omega\} \subseteq \text{acl}(A \cup \{a\})$ so for each n we can find $k_n < \omega, \bar{c}_n \in {}^\omega A$ and a formula $\varphi(x, y, \bar{z})$ such that $\mathfrak{C} \models \varphi(b_0^n, a, \bar{c}_n)$ & $(\exists^{\leq k_n} x) \varphi(x, a, \bar{c}_n)$.

Let $\bar{a}_\alpha^n = \langle b_\alpha^n \rangle^{\wedge} \bar{c}_n$ and φ_n, k_n have already been chosen.

Now check Definition 3.2. □_{3.6}

3.7 Definition. We say that $\vartheta(x_1, x_2; \bar{c})$ is a finite-to-finite function from $\varphi_1(\mathfrak{C}, \bar{a}_2)$ onto $\varphi_2(\mathfrak{C}, \bar{a}_2)$ if:

- (a) if $b_2 \in \varphi_2(\mathfrak{C}, \bar{a}_2)$ then the set $\{x : \vartheta(x_1, b_2, \bar{c}) \wedge \varphi_1(x, \bar{a}_2)\}$ satisfies:
 - (i) it is finite but
 - (ii) it is not empty except for finitely many such b_2 's
- (b) if $b_1 \in \varphi_1(\mathfrak{C}, \bar{a}_2)$ then the set $\{x : \vartheta(b_1, x, \bar{c}) \wedge \varphi_2(x, \bar{a}_2)\}$ satisfies:
 - (i) it is finite but:
 - (ii) it is not empty except for finitely many such b_1 's.

3.8 Claim. *If T is strongly dependent then the following are impossible:*

$(St)_1$ for some $\varphi(x, \bar{a})$

- (a) $\varphi(x, \bar{a})$ is not algebraic
- (b) E is a definable equivalence relation (in \mathfrak{C} by a first order formula possible with parameters) with domain $\subseteq \varphi(\mathfrak{C}, \bar{a}_1)$ and infinitely many equivalence classes
- (c) there is a formula $\vartheta(x, y)$ such that for every $b \in \text{Dom}(E)$ for some \bar{c} , the formula $\vartheta(x, y; \bar{c})$ is a finite to finite map from $\varphi(\mathfrak{C}, \bar{a})$ into b/E ;

$(St)_2$ for some formulas $\varphi(x), xEy, \vartheta(x, y, z)$ possibly with parameters we have:

- (a) $\varphi(x)$ is non-algebraic
- (b) $xEy \rightarrow \varphi(x) \ \& \ \varphi(y)$
- (c) for uncountably many $c \in \varphi(\mathfrak{C})$ the formula $\vartheta(x, y; c)$ is a finite to finite function from $\varphi(x)$ into xEc
- (d) for some $k < \omega$, if $b_1, \dots, b_k \in \varphi(\mathfrak{C})$ are pairwise distinct then $\bigwedge_{\ell=1}^k xEb_\ell$ is algebraic

$(St)_3$ similarly with $\varphi(x, \bar{a})$ replaced by a type, as well as xEy .

Proof. We concentrate on $(St)_2$; the others are similar.

Let $\bar{z}_n = (z_0^2, z_1^1, z_1^2, z_2^1, z_2^2, z_3^1, z_3^2, \dots, z_{n-1}^1, z_{n-1}^2)$. We shall now define by induction on $n < \omega$ formulas $E_n(x, y, \bar{z}_n)$ and $\vartheta_n(x_1, x_2, \bar{z}_n)$ also written $E_{\bar{z}_n}^n(x, y), \vartheta_{\bar{z}_n}^n(x, y)$.

Case 1: $n = 0$.

So $(\bar{z}_n = \langle \rangle)$, and $E_n(x, y) = \varphi(x) \ \& \ \varphi(y)$,
 $\vartheta_n(x_1, x_2) = (x_1 = x_2)$.

Case 2: $n = m + 1$.

Let $E_{\bar{z}_n}^n(x, y) =: E_{\bar{z}_m}^m(x, z_m^1) \ \& \ E_{\bar{z}_m}^m(y, z_m^1) \ \& \ (\exists x', y')[x'Ey' \wedge \vartheta_{\bar{z}_m}^m(x', x, z_{n-1}) \ \& \ \vartheta_{\bar{z}_m}^m(y', y, z_n)]$,

$\vartheta_n(x_1, x_2, \bar{z}_n) = E_{\bar{z}_m}^m(x_2, x_2) \ \& \ \varphi(x_1, \bar{a}) \ \& \ (\exists x')[\vartheta(x_1, x') \ \& \ \vartheta_m(x', x_2, \bar{z}_m)]$.

We now prove by induction on n , letting $\psi_n(\bar{z}_n) = z_m E_{\bar{z}_m}^m z_m$ that:

$(*)$ (a) if $\psi_n(\bar{z}_n)$ then

- (α) $E_{\bar{z}_n}^n$ is a two-place relation with domain non empty subset of $\varphi(\mathfrak{C}, \bar{a})$
- (β) $\vartheta_{\bar{z}_n}^n(x_1, x_2)$ is a finite to finite function from a co-finite subset of $\varphi(\mathfrak{C}, \bar{a})$ onto $\text{Dom}(E_{\bar{z}_n}^n)$

- (γ) if $m < n$ then the domain of $E_{\bar{z}_n}^n$ is included in one set of the form $\{x : xE_{\bar{z}_m}^m y\}$

(b)(α) $\psi_0(<>)$ holds

(β) if $\psi_0(\bar{z}_n)$ and $z_n \in \text{Dom}(E_{\bar{z}_n}^n)$ then $\psi_{n+1}(\bar{z}_n \hat{\ } \langle z_n \rangle)$ holds.

This is straight.

We can choose $b_\alpha \in \varphi(\mathfrak{C}, \bar{a})$ for $\alpha < \omega_1$ pairwise distinct such that each formula xEb_α is not algebraic and $\vartheta(x, y; b_\alpha)$ is a finite-to-finite function from $\varphi(\mathfrak{C}, \bar{a})$ into (the formula) $x \in b_\alpha$. Without loss of generality the finite number implicit in Definition 3.7 for b_α does not depend on α . Without loss of generality $\langle b_\alpha : \alpha < \lambda \rangle$ is an indiscernible sequence, $\lambda \geq \omega_1$. Let $b_\alpha^\ell = b_{2\alpha+\ell}$ for $\alpha < \lambda, \ell < 2$.

Now for every sequence $\bar{\alpha} = \langle \alpha_n : n < \omega \rangle$ such that $n < \omega \Rightarrow \alpha_n < \alpha_{n+1} < \lambda$ we consider $\bar{b}_\alpha^n = \bar{b}_{\alpha \upharpoonright n}^n = \langle (b_{\alpha_\ell}^1, b_{\alpha_\ell}^2) : \ell < n \rangle$ and $p_{\bar{\alpha}} = \{\psi_n(x, \bar{b}_\alpha^n) : n < \omega\}$; note that x gets the “place” of z_0^2 . Now

\otimes_1 for $\bar{\alpha}$ above $p_{\bar{\alpha}}$ is consistent.

[Why? We choose $a_{\bar{\alpha}}^n$ realizing $p_{\bar{\alpha} \upharpoonright n}$ for $\bar{\alpha} \in {}^n(\omega_1)$ by induction on n]

\otimes_2 if c realizes $p_{\bar{\alpha}}$ then for every n , $b_{\alpha_n}^1$ is algebraic over $\{c, b_{\alpha_0}^1, b_{\alpha_0}^2, \dots, b_{\alpha_{n-1}}^1, b_{\alpha_{n-1}}^2\}$.

[Why? By $(*)(a)(\beta)$.]

This is more than enough to show T is not strongly dependent. $\square_{3.8}$

Discussion: Phrase 3.8 for ideals of small formulas.

3.9 Claim. *If T is strongly dependent, then \otimes_ℓ is impossible for $\ell = 1, 2, 3, 4$ where:*

- $\otimes_1(a)$ $\langle \bar{a}_\alpha : \alpha < \lambda \rangle$ is an indiscernible sequence over A
 - (b) $u_n \subseteq \lambda$ is finite, [non-empty] with $\langle u_n : n < \omega \rangle$ having pairwise disjoint convex hull for $n \neq m$
 - (c) $\bar{b} \in {}^{\omega>} \mathfrak{C}$
 - (d) for each n for some α_n, k and $t_{n(0)}^{\mathbf{t}} < \dots < t_{n(k-1)}^{\mathbf{t}} \in u_n$ for $\mathbf{t} \in \{\text{false}, \text{truth}\}$ and $\bar{c}_n \in {}^{\omega>} A$ and φ we have $\mathfrak{C} \models \varphi(\bar{c}, a_{t_{n(0)}}, \dots, a_{t_{n(k-1)}}, \bar{c}_n)^{\mathbf{t}}$ for both values of \mathbf{t}
- \otimes_2 like \otimes_1 but allows \bar{a}_α to be infinite (so without loss of generality each u_n is a singleton)
- $\otimes_3(a)$ $\langle \bar{a}_\alpha^n : \alpha < \lambda \rangle$ is an indiscernible sequence over $A \cup \{a_\beta^m : m < \omega \ \& \ m \neq n \text{ and } \beta < \lambda\}$
 - (b) $a_\alpha^n \neq a_{\alpha+1}^n$

- (c) *some $a \in \mathfrak{C}$ satisfies $n < \omega \Rightarrow \text{acl}(A \cup \{a\}) \cap \{a_\alpha^n : \alpha < \lambda\} \neq \emptyset$*
- \otimes_4 *like \otimes_3 but (d)' for some $a \in \mathfrak{C}$ for every n the sequence $\langle a_\alpha^n : \alpha < \lambda \rangle$ is not an indiscernible sequence over $A \cup \{a_\beta^m : m \neq n, \beta < \lambda\}$.*

Proof. Similar to the previous ones.

3.10 Question: Show that the theory of the p -adic field is strongly dependent.

§4 DEFINABLE GROUPS

4.1 Context.

- (a) T is (first order complete)
- (b) \mathfrak{C} is a monster model of T .

We try here to generalize the theorem on the existence of commutative infinite subgroups to T with the dependence property. The theorems on definable groups in a monster \mathfrak{C} , $\text{Th}(\mathfrak{C})$ stable, are well known.

4.2 Definition. G is definable in \mathfrak{C} means that: a finite type $p^G(x)$ with parameters from B_2^G by defining the set of elements of G , and the operators $xy = z$ and $x^{-1}y$ are definables by the types $p_2^G(x, y, z), p_3^G(x, b)$ hence by such formulas; (the operation are written ab, a^{-1} ; no confusion arise as we have just one group) with parameters from B_1^G .

4.3 Claim. Assume

- (a) T has the dependence property
- (b) G is a type definable group in \mathfrak{C}
- (c) $A \subseteq G$ is a set of pairwise commuting elements, D a non-principal ultrafilter on A or just
- (c)⁻ $A \subseteq G, D$ a non-principal ultrafilter on A such that $(\forall^D a_1)(\forall^D a_2)(a_1 a_2 = a_2 a_1)$.

Then there is a formula $\varphi(x, \bar{a})$ such that:

- (α) $\varphi(x, \bar{a}) \in \text{Av}(\bar{a}, D)$
- (β) $G \cap \varphi(\mathfrak{C}, \bar{a})$ is an abelian subgroup of G
- (γ) $\bar{a} \subseteq A \cup B_2^G \cup \{c : c \text{ realizes } \text{Av}(A \cup B_2^G, D)\}$.

Remark. 1) If D is a principal ultrafilter say $\{a^*\} \in D$ then $\varphi(x, \bar{a})$ is essentially $\text{Cm}_G(\text{Cm}_G(a^*))$ so no new point.

2) If D is a nonprincipal ultrafilter, then necessarily $\varphi(x, \bar{a})$ is not algebraic as it belongs to $\text{Av}(\bar{a}, D)$.

Proof. We try to choose a_n, b_n by induction on $n < \omega$ such that:

- (i) a_n, b_n realizes $p_n = p^G(x) \cup \text{Av}(A_n, D)$ where $A_n = A \cup B_2^G \cup \{a_k, b_k : k < n\}$
- (ii) a_n, b_n does not commute (in G , they are in G because $p^G(x) \subseteq p_n$).

Case 1: We succeed.

For $n < m < \omega$, $c' \in \{a_n, b_n\}$ and $c'' \in \{a_m, b_m\}$ clearly c', c'' are in G and they commute (as c'' realizes $\text{Av}(A \cup B_2^G \cup \{c'\}, D)$ and c' realizes $\text{Av}(A \cup B_2^G, D)$ and assumption (c) about commuting in A or by clause $(c)^-$). Hence if $k < \omega$, $n_0 < \dots < n_{k-1} < \omega$ then $c =: b_{n_0} b_{n_1} \dots b_{n_{k-1}}$ satisfies: c, b_n commute iff $n \notin \{n_0, \dots, n_{k-1}\}$, so $\varphi(x, y) = [xy = yx]$ has the independence property contradiction to assumption (a).

Case 2: We are stuck at $n < \omega$.

So $p_n(x) \cup p_n(y) \vdash (xy = yx)$, hence there are formulas $\psi_0(x, \bar{b}_0) \in p^G(x)$ and $\varphi(x, \bar{a}_0) \in \text{Av}(A_m, D)$ and letting $\psi(x, \bar{a}^*) = [\psi_0(x, \bar{b}_0) \wedge \varphi(\bar{x}, \bar{a})]$ we have

$$(*)_1 \quad \psi(x, \bar{a}^*) \ \& \ \psi(y, \bar{a}^*) \vdash xy = yx \text{ (so both products are well defined).}$$

As p^G is a formula, without loss of generality $p^G(x) = \{\psi_0(\bar{x}, \bar{b}_0)\}$ and so $\psi(x, \bar{a}) \vdash p^G(x)$. Hence $\psi(\mathfrak{C}, \bar{a})$ is a set of pairwise commuting elements of G , so $\text{Cm}_G(\text{Cm}_G(\psi(\mathfrak{C}, \bar{a})))$ is an abelian subgroup of G containing $\psi(\mathfrak{C}, \bar{a})$ and is definable, with parameters as required in clause (γ) .

[Formally let $\vartheta(x, \bar{a}) = (\forall y)(\theta(y, \bar{a}) \rightarrow (yx = xy))$ (and both well defined)). So $\psi(x, \bar{a}) \vdash \vartheta_1(x, \bar{a})$.

Let

$$\varphi(x) = \varphi(x, \bar{a}) = \theta(x, \bar{a}) \ \& \ (\forall y)[\vartheta(y, \bar{a}) \rightarrow xy = yx \text{ (both well defined)}].$$

So $\psi(x, \bar{a}) \vdash \varphi(x, \bar{a}) \vdash \vartheta(x, \bar{a})$ hence $\varphi(x, \bar{a}) \in p_n = \text{Av}(a_n, D)$ hence $\varphi(x, \bar{a}) \in \text{Av}(\bar{a}, D)$ and \bar{a} is $\subseteq A_n \subseteq A \cup B^G \cup \cup \{\bar{c} : \bar{c} \text{ realizes } \text{Av}(A \cup B^G, D)\}$.

Lastly, $\varphi(\mathfrak{C}, \bar{a}) \cap G$ is abelian by the definition of $\varphi(x)$ as $\varphi(x, \bar{a}) \vdash \vartheta(x, \bar{a})$.]

□_{4.3}

4.4 Claim. *Assume*

- (a) G is a definable (infinite) group, see 4.3
- (b) every element of G commutes only with finitely many others.

Then T is not strongly dependent.

Proof. Assume first $p^G = \{\varphi(x)\}$.

Let $xEy = [x, y \text{ are conjugates}]$, i.e.

$$\vartheta(x_1, x_2, y) = (x_1 = yx_2y^{-1}).$$

We shall get a contradiction by 3.8. Now $\varphi(x), \vartheta(x_1, x_2, y)$ satisfies the demands in $(St)_1$ there, which is impossible if T is strongly dependent so we are done.

If p^G is a type use $(St)_3$ of 3.8. $\square_{4.4}$

4.5 Context.

- (a) T is (first order complete)
- (b) \mathfrak{C} is a monster model of T .

4.6 Definition. 1) We say G is a type-group (in \mathfrak{C}) if $G = (p, *, \text{inv}) = (p^G, *^G, \text{inv}^G)$ where

- (a) $p = p(x)$ is a type
- (b) $*$ is a two-place function on \mathfrak{C} , possibly partial, definable (in \mathfrak{C}) with parameters from $\text{Dom}(p)$, we normally write ab instead of $a * b$ or $*(a, b)$
- (c) $(p(\mathfrak{C}), *)$ is a group, we write $x \in G$ for $x \in p(\mathfrak{C})$
- (d) inv^G is a (partial) unary function, definable (in \mathfrak{C}) with parameters from $\text{Dom}(p)$, which on $p(\mathfrak{C})$ is the inverse, so if no confusion arise we shall write $(x)^{-1}$ for $\text{inv}(x)$.

2) We say G is a definable group if $p(*)$ is a formula, i.e., a singleton.

3) We say G is an almost type definable group if $p(x)$ is replaced by $\bar{p} = \langle p_i(\bar{x}) : i < \delta \rangle, p_i(\bar{x})$ decreasing with i .

Discussion: 1) We can consider $(p, \mathbf{p}, *)$, p a type $p \subseteq \mathbf{p} \in \mathcal{S}(\mathfrak{C})$ and $*$ written $a * b$ is a partial two place function defined with parameters from $\text{Dom}(G)$ (and inv) such that:

- (a) if $a \in p(\mathfrak{C}), b$ realizes $\mathbf{p} \upharpoonright (\text{Dom}(p)) \cup \{a\}$ then $a * b$ is well defined
- (b) $(a * b) * c = a * (b * c)$.

Interesting?

2) We may restrict ourselves to $\mathbf{p} = \text{Av}(B, D), p = \text{Av}(B, D), D$ an ultrafilter on B .

3) We may consider the situations defined below forgetting the group.

4.7 Definition. 1) A place \mathbf{p} is a tuple $(p, B, D, *, \text{inv}) = (p[\mathbf{p}], *_{\mathbf{p}}, \text{inv}_{\mathbf{p}})$ such that:

- (a) B is a set, D is an ultrafilter on B , $p \subseteq \text{Av}(B, D)$
- (b) $*$ is a partial two-place function defined with parameters from B ; we shall write $a *_{\mathbf{p}} b$ or when clear from the content $a * b$ or ab
- (c) inv is a partial unary function definable from parameters in B .

1A) \mathbf{p} is non trivial if $\text{Av}(A, D)$ is not algebraic for every A .

2) We say \mathbf{p} is weakly a place in a definable group G or type definable group G if (\mathbf{p} is a place, $p^{\mathbf{p}} \vdash p^G$, the set $B^{\mathbf{p}}$ includes $\text{Dom}(p^G)$ and the operations agree on $p^{\mathbf{p}}[\mathfrak{C}]$ when the place operations are defined.

2A) If those operations are the same, we say that \mathbf{p} is strongly a place in G .

3) We say $\mathbf{p}_1 \leq \mathbf{p}_2$ if both are places, $B^{\mathbf{p}_1} \subseteq B^{\mathbf{p}_2}$ and $p[\mathbf{q}_1] \vdash p[\mathbf{p}_1]$ and the operations are same.

4) $\mathbf{p} \leq_{\text{dir}} \mathbf{q}$ if $\mathbf{p} \leq \mathbf{q}$ and $B^{\mathbf{q}} \subseteq A \Rightarrow \text{Av}(A, D^{\mathbf{p}}) = \text{Av}(A, D^{\mathbf{q}})$.

4.8 Definition. 1) A place \mathbf{p} if σ -closed if:

- (a) σ has the form $\sigma(\bar{x}_1; \dots; \bar{x}_{n(*)})$, a term in the vocabulary of groups
- (b) if $\bar{a}_n \in {}^{(\ell g(\bar{x}_n))}\mathfrak{C}$, for $\ell \leq n$ and $B \subseteq A$, then¹ $\sigma(\bar{a}_1, \dots, \bar{a}_{n(*)})$ realizes $\text{Av}(A \cup \bar{a}_1 \hat{\ } \dots \hat{\ } \bar{a}_{n(*)}, D)$ provided that
 - (*) $n \leq n(*)$ & $\ell < \ell g(\bar{a}_n) \Rightarrow a_{n,\ell}$ realizes $\text{Av}(A \cup \bar{a}_0 \hat{\ } \bar{b}_0 \hat{\ } \dots \hat{\ } \bar{a}_{n-1}, D)$.

2) A place \mathbf{p} is $(\sigma_1 = \sigma_2)$ -good or satisfies $(\sigma_1 = \sigma_2)$ if

- (a) $\sigma_\ell = \sigma_\ell(\bar{x}_1, \dots, \bar{x}_{n(*)})$ a term in the vocabulary of group for $\ell = 1, 2$
- (b) if $\bar{a}_n \in {}^{(\ell g(\bar{x}_n))}\mathfrak{C}$ for $\ell \leq n$ then $\sigma_1(\bar{a}_1; \dots; \bar{a}_{n(*)}) = \sigma_2(\bar{a}_0; \dots; \bar{a}_{n(*)})$ then $\sigma_1(\bar{a}_1; \dots; \bar{a}_{n(*)}) = \sigma_2(\bar{a}_1, \dots, \bar{a}_{n(*)})$ whenever (*) of part (2) holds for $A = B$.

3) We can replace σ in part (1) by a set of terms. Similarly in part (2).

4) We may write x_ℓ instead of $\langle \ell \rangle$. So if we write $\sigma(\bar{x}_1; \bar{x}_2) = \sigma(x_1; x_2) = x_1, x_2$ or $\sigma = x_1, x_2$ we mean $x_1 = x_{1,0}, x_2 = x_{2,0}, \bar{x}_1 = \langle x_{1,0} \rangle, \bar{x}_2 = \langle x_{2,0} \rangle$. We may use also $\sigma(\bar{x}; \bar{y})$ instead of $\sigma(\bar{x}_1; \bar{x}_2)$ and $\sigma(\bar{x}; \bar{y}; \bar{z})$ similarly.

¹so all the stages in the computation of $\sigma(\bar{a}_0; \dots; \bar{a}_{n(*)})$ should be well defined

- 4.9 Definition.** 1) We say a place \mathbf{p} is a poor semi-group if it is σ -closed for $\sigma = xy$ and satisfies $(x_1x_2)x_3 = x_1(x_2x_3)$.
 2) We say a place \mathbf{p} is a poor group if it is a poor semi group and is σ -closed for $\sigma = (x_1)^{-1}x_2$.
 3) We say a place \mathbf{p} is a quasi semi group if for any semi group term $\sigma_*(\bar{x})$, \mathbf{p} is σ -closed for $\sigma(\bar{x}; y) = \sigma_*(\bar{x})y$.
 4) We say a place \mathbf{p} is a quasi group if for any group term $\sigma_1(\bar{x}), \sigma_2(\bar{x})$, \mathbf{p} is σ -closed for $\sigma(\bar{x}; y) = \sigma_1(\bar{x})y\sigma_2(\bar{x})$.
 5) We say \mathbf{p} is abelian if it is (xy) -closed and satisfies $xy = yx$.
 6) We say \mathbf{p} is affine if \mathbf{p} is $(xy^{-1}z)$ -closed.

4.10 Definition. 1) We say a place \mathbf{p} is a group if $G = G^{\mathbf{p}} = (\text{Av}(B^{\mathbf{p}}, D), {}^*\mathbf{p}, \text{inv}_{\mathbf{p}})$ is a group.

- 4.11 Claim.** 1) *The obvious implications hold.*
 2) *If we use $\bar{\mathbf{b}}$ every $\bar{\mathbf{b}}'$ realizing the same type has the same properties.*

* * *

We now note that there are places

4.12 Claim. 1) *Assume that \mathbf{p} is a group and $a_n \in p^G[\mathfrak{C}]$ for $n < \omega$. We define $a_{[u]} \in p^{\mathbf{p}}[\mathfrak{C}]$ for any finite non-empty $u \subseteq w$ by induction on $|u|$, if $u = \{n\}$ then $a_{[u]} = a_n$, if $|u| > 1$, $\max(u) = n$ then $a_{[u]} = a_{[u \setminus \{n\}]} * a_n$ (and we are assuming they are all well defined and $a_{[u_1]} \neq a_{[u_2]}$ when $u_1 \triangleleft u_2$. Then we can find D^*, \mathbf{q} such that:*

- (a) \mathbf{q} is a place inside G
- (b) \mathbf{q} is a poor semi group and non trivial
- (c) $B^{\mathbf{q}} = B^G \cup \cup \{a_{[u]} : u \subseteq [w]^{<\aleph_0}\}$
- (d) D^* is an ultrafilter on $[w]^{<\aleph_0}$ such that $(\forall n[w \setminus n]^{\aleph_0} \in D^* \text{ and})$ for every $Y \in D^*$ we can find $Y' \subseteq Y$ from D^* closed under union (of two)
- (e) \bar{b} is an indiscernible sequence based on $(B^G \cup \{a_{[u]} : u \in [w]^{<\aleph_0}, u \neq \emptyset\}, D^{\mathbf{q}})$
- (f) $D^{\mathbf{q}} = \{a_{[u]} : u \in Y\} : Y \in D^*\}$
- (g) *if the a_n 's commute (i.e. $a_na_m = a_ma_n$ for $n \neq m$) then \mathbf{q} is Abelian.*

Proof. By a well known theorem of Glazer, relative of Hindman theorem saying D^* as in (e) exists.

4.13 Claim. *Assume*

- (a) \mathbf{p} is a place in a type definable group (or much less)
- (b) the place \mathbf{p} is semi group
- (c) \mathbf{p} is commutative (in the sense of Definition 4.8, so satisfies $x * Y = y * x$ but not necessarily $x_{1,0} \neq x_{1,1} = x_{1,1} * x_{1,0}$)
- (d) if $A \supseteq B^{\mathbf{p}}$ then for some b, c realizing $\text{Av}(D^{\mathbf{p}}, A)$, $c *_G b, b *_G c$ are (necessarily well defined, and) distinct.

Then T has the independence property.

Remark. This is related to the well known theorems on stable theories (see Zilber and Hrushovski's work).

Proof. We choose A_i, b_i, c_i by induction on $i < \omega$.

In stage i first let $A_i = B^{\mathbf{p}} \cup \{b_j, c_j : j < i\}$ and add B^G if $B^G \not\subseteq B^{\mathbf{p}}$.
Second, choose b_i, c_i realizing $\text{Av}(A_i, D^{\mathbf{p}})$ such that $b_i * c_i \neq c_i * b_i$.

Now if $i < j < \omega$ any $a' \in \{b_i, c_i\}, a'' \in \{b_j, c_j\}$ then a' realizes $\text{Av}(A_i, D^{\mathbf{p}})$ and a'' realizes $\text{Av}(A_j, D^{\mathbf{p}})$ which include $\text{Av}(A_i \cup \{a'\}, D^{\mathbf{p}})$. So by assumption (c), the elements a', a'' commute in G .

So as in well known, for $n < \omega, i_0 < i_1 < \dots < i_n$ the element $b_{i_0} * b_{i_1} * \dots * b_{i_{n-1}}$ commute in G with a_j iff $j \notin \{i_0, \dots, i_{n-1}\}$ hence T has the independence property.

□_{4.13}

Note that 4.14 is interesting for G with a finite bound on the order of elements as if $a \in G$ has infinite order then $GM_G(Gm_G(a))$ is as desired.

4.14 Conclusion. [T is dependent].

Assume G is a definable group.

- 1) If \mathbf{p} is a commutative semi group in G , non trivial, then for some formula $\varphi(x, \bar{a})$ such that $\varphi(\bar{x}) \vdash "x \in G", \varphi(x, \bar{a}) \in \text{Av}(\bar{a}, D^{\mathbf{p}})$ and $G \upharpoonright \varphi(\mathfrak{C})$ is a commutative group.
- 2) If G has an infinite abelian subgroup, then it has an infinite definable commutative subgroup.

Proof. 1) By 4.13 for some $A \supseteq B^{\mathbf{p}}$ for every b, c realizing $q = \text{Av}(A, D^{\mathbf{p}})$ we have $G \models cb \neq bc$, i.e., the elements of $q(\mathfrak{C})$, which are all in G , pairwise commute. By compactness there is a formula $\varphi_1(x) \in p[\mathbf{p}]$ such that the elements of $\varphi_1(\mathfrak{C}) \cap G$ pairwise commute and without loss of generality $\varphi_1(x) \vdash [x \in G]$ but this set is not

necessarily a subgroup. Let $\varphi_2(x) =: [x \in G] \ \& \ (\forall y)(y, \varphi_1(y) \rightarrow x * y = y * x)$. Clearly $\varphi_1(\mathfrak{C}) \subseteq \varphi_2(\mathfrak{C}) \subseteq G$ and every member of $\varphi_2(\mathfrak{C})$ commute with every member of $\varphi_1(\mathfrak{C})$. So $\varphi(z) =: [z \in G] \wedge (\forall y)[\varphi_2(y) \rightarrow yz = xy]$ is first order and define the center of $G \upharpoonright \varphi_2[\mathfrak{C}]$ which includes $\varphi_1(\mathfrak{C})$, so we are done.

2) Let $G' \subseteq G$ be infinite abelian. Choose by induction on $n < \omega$, $a_n \in G'$ as required in 4.13. $\square_{4.14}$

4.15 Remark. So 4.14 tells us that having some commutativity implies having alot. If in 4.13 every $a_{[u]}$ is not a “small” definable set defined with parameters in $B^{\mathbf{P}} \cup \{a_n : n < \max(u)\}$, then also $\varphi(x, \bar{a})$ is not small where small means some reasonably definable ideal.

* * *

4.16 Definition. Assume

- (a) G is a type definable semi group
- (b) $M \supseteq B^G$ is $(|T| + |B^G|)^+$ -saturated
- (c) \mathfrak{D} is the set of ultrafilters D on M such that $p^G \subseteq \text{Av}(M, D)$
- (d) on $\mathfrak{D} = \mathfrak{D}_{G,M}$ we define an operation

$D_1 * D_2 = D_3$ iff for any $A \supseteq$ and a realizing

$\text{Av}(A, D_1)$ and b realizing $\text{Av}(A + a, D_2)$ the element
 $a * b$ realizes $\text{Av}(A, D_3)$.

- (e) $ID_{G,M} = \{D \in \mathfrak{D}_{G,M} : D * D = D\}$
- (f) $H_{G,M}^{\text{left}} = \{a \in G : \text{for every } D \in \mathfrak{D}, \text{ and } A \text{ if } b \text{ realizes } \text{Av}(A + a, D) \text{ and } a * b \text{ realizes } \text{Av}(A, D)\}$
- (g) $H_{G,M}^{\text{right}}$ similarly and $H_{G,M}$ their intersection.

It is known.

4.17 Fact: \mathfrak{D} is a semi group, i.e., associativity holds the operation is continuous in the second variable hence then is an idempotent (even every subset closed under $*$ and topologically $\neq \emptyset$ has an idempotent).

Note

4.18 Fact: 1) If G is a group, then

- (a) $H_{G,M}^{\text{left}}$ is a subgroup of G , with bounded index, is of the form $\{q(\mathfrak{C}) : q \in \mathcal{S}_{G,M}^{\text{left}} \subseteq S(M)\}$
 - (b) Similarly $H_{G,M}^{\text{right}}, H_{G,M}, H_{G,M}^{\text{right}} \cap H_{G,M}^{\text{left}}$ with $\mathcal{S}_{G,M}^{\text{right}}, \mathcal{S}_{G,M}$.
- 2) If $D \in \mathfrak{D}$ is non principal and $\text{Av}(M, D) \in \mathcal{S}_{G,M}^{\text{right}}$, then for any $A \supseteq M$ and element a realizing $\text{Av}(A, D)$ and b realizing $\text{Av}(A + a, D)$ we have
- (α) $a *_G b$ realizes $\text{Av}(A, D)$?
 - (β) also $a^{-1} * b \in D$.
- 3) $\mathcal{S}_{G,M}^{\text{left}} \subseteq ID_{G,M}$.
- 4) Similarly for $\mathcal{S}_{G,M}^{\text{left}}, b *_G a$.
- 5) If $D \in \mathfrak{D}, p = \text{Av}(M, D) \in \mathcal{S}_{G,M}$ then
- (a) $\mathbf{p} = (M, D, *, \text{inv})$ is a quasi group
 - (b) $\{a^{-1}b : a, b \in p(M)\}$ is a subgroup of G with bound index, in fact is $\{a \in \mathfrak{C} : \text{tp}(a, M) \in \mathcal{S}_{G,M}\}$.

§5 NON-FORKING

5.1 *Hypothesis.* T is dependent.

5.2 Definition. [Sh:93] 1) An α -type $p = p(\bar{x})$ divides over B if some $\bar{\mathbf{b}}, \varphi(\bar{x}, \bar{y})$ witness it which means

- (a) $\bar{\mathbf{b}} = \langle \bar{b}_n : n < \omega \rangle$ is an indiscernible sequence over B
- (b) $\varphi(\bar{x}, \bar{y})$ is a formula with $\ell g(\bar{y}) = \ell g(\bar{b}_n)$

such that

- (c) $p \vdash \varphi(\bar{x}, \bar{b}_0)$
- (d) $\{\varphi(\bar{x}, \bar{b}_n) : n < \omega\}$ is contradictory.

2) An α -type p forks over B if for some $\langle \varphi_\ell(\bar{x}, \bar{a}_\ell) : \ell < k \rangle$ we have $p \vdash \bigvee_{\ell < k} \varphi_\ell(\bar{x}, \bar{a}_\ell)$

and $\{\varphi_\ell(\bar{x}, \bar{a}_\ell)\}$ divides over B for each $\ell < k$ (note: though \bar{x} may be infinite the formulas are finitary).

3) We say C/A does not fork over B if letting $\bar{\mathbf{c}}$ list C , $\text{tp}(\bar{\mathbf{c}}, A)$ does not fork over B , or what is equivalent $\bar{\mathbf{c}} \in {}^\omega C \Rightarrow \text{tp}(\bar{\mathbf{c}}, A)$ does not fork over B (so below we may write claims for $\bar{\mathbf{c}}$ and use for C).

4) The m -type p is f.s. (finitely satisfiable) in A if every finite $q \subseteq p$ is realized by some $\bar{b} \subseteq A$.

5) The Δ -multiplicity of p over B is $\text{Mult}_\Delta(p, B) = \sup\{|\{q \upharpoonright \Delta : p \subseteq q \in \mathcal{S}^m(M), q \text{ does not fork over } B\}| : M \supseteq B \cup \text{Dom}(p)\}$.

Omitting Δ means $\mathbb{L}(\tau_T)$, omitting B we mean $\text{Dom}(p)$.

5.3 Definition. 1) Let $p = p(\bar{x})$ be an α -type and Δ be a set of $\mathbb{L}(T)$ -formulas of the form $\varphi(\bar{x}, \bar{y})$ and $k \leq \omega$. For a type $p(\bar{x})$ we say that it (Δ, k) -divides over A if some $\bar{\mathbf{b}}, \varphi(\bar{x}, \bar{y})$ witness it which means

- (a) $\bar{\mathbf{b}} = \langle \bar{b}_n : n < 2k + 1 \rangle$ is Δ -indiscernible
- (b) $\varphi(\bar{x}, \bar{y}) \in \mathbb{L}(\tau_T)$
- (c) $p \vdash \varphi(\bar{x}, \bar{b}_0)$
- (d) $\{\varphi(\bar{x}, \bar{b}_n) : n < 2k + 1\}$ is k -contradictory.

2) For a type $p(\bar{x})$ we say that it (Δ, k) -forks over B if $p \vdash \bigvee_{\ell < n} \varphi_\ell(\bar{x}, \bar{a}_\ell)$ for some $n, \varphi_\ell(\bar{x}, \bar{y})$ and \bar{a}_ℓ , where each $\varphi_\ell(\bar{x}, \bar{a}_\ell)(\Delta, k)$ -fork over B .

5.4 Observation: 0) In Definition 5.2(1), if $p = \{\varphi(x, \bar{b})\}$ then without loss of generality $\bar{b} = \bar{b}_0$. If p divides over B then p forks over B .

0A) Forking is preserved by permuting and repeating the variables. If $\text{tp}(\bar{b} \hat{\ } \bar{c}, A)$ does not fork over B then so does $\text{tp}(\bar{b}, A)$ and both does not divide over B . Similarly for dividing.

1) If $p \in \mathcal{S}^m(A)$ is finitely satisfiable in B , then p does not fork over B ; hence every type over M does not fork over M .

2) If $p \in \mathcal{S}^m(A)$ does not fork or just does not divide over $B \subseteq A$, then p does not split strongly over B .

3) [extension property] If an m -type p is over A and does not fork over B , then some extensions $q \in \mathcal{S}(A)$ of p does not fork over B .

4) [few nonforking types] For $B \subseteq A$ the set $\{p \in \mathcal{S}^m(A) : p \text{ does not fork over } B\}$ has cardinality $\leq 2^{2^{|B|+|T|}}$.

5) [monotonicity] If $B_1 \subseteq B_2 \subseteq A_2 \subseteq A_1$ and $p \in \mathcal{S}(A_1)$ does not fork over B_1 , then $p \upharpoonright A_2$ does not fork over B_2 .

6) [indiscernible preservation] If \bar{b} is an infinite indiscernible sequence over A_1 and $B \subseteq A_1 \subseteq A_2$ and $\bar{b} \subseteq A_2$ and $\text{tp}(\bar{c}, A_2)$ does not fork over B or just does not divide over B then \bar{b} is an (infinite) indiscernible sequence over $A_1 \cup \bar{c}$.

7) [finite character] If p forks over B then some finite $q \subseteq p$ does; if p is closed under conjunction (up to equivalent suffice) then we can choose $q = \{\varphi\}$. Similarly for divides.

8) [monotonicity in the type] If $p(\bar{x}) \subseteq q(\bar{x})$ or just $q(\bar{x}) \vdash p(\bar{x})$ and $p(\bar{x})$ forks over B then $q(\bar{x})$ forks over B ; similarly for divides.

9) An m -type p is finitely satisfiable in A iff for some ultrafilter D on ${}^m A$ we have $p \subseteq \text{Av}(\text{Dom}(p), D)$.

10) If M is $|B|^+$ -saturated and $B \subseteq M$ and $p \in \mathcal{S}^n(M)$ then p does not fork over B iff p does not strongly split over B .

Remark. 1) Only parts (2), (4), (6) of 5.2 use “ T is dependent”; (4) as in [Sh 3].

2) If T is unstable then for every κ there are for some A and $p \in \mathcal{S}(A)$ such that p divides over every $B \subseteq A$ of cardinality $< \kappa$ (use a Dedekind cut with both cofinalities $\geq \kappa$).

Proof. 0), 0A), 1) Easy.

2) Assume toward contradiction that p splits strong, then for some infinite indiscernible sequence $\langle \bar{b}_n : n < \omega \rangle$ over B and $n < m$ we have $[\varphi(\bar{x}, \bar{b}_n) \equiv \neg \varphi(\bar{x}, \bar{b}_m)] \in p$ really $p \vdash [\varphi(\bar{x}, \bar{b}_n) \equiv \neg \varphi(\bar{x}, \bar{b}_m)]$ suffice. By renaming, without loss of generality $n = 0, m = 1$. Let $\bar{c}_n = \bar{b}_{2n} \hat{\ } \bar{b}_{2n+1}$, $\psi(\bar{x}, \bar{c}_n) = [\varphi(\bar{x}, \bar{b}_{2n}) \equiv \neg \varphi(\bar{x}, \bar{b}_{2n+1})]$. Clearly $\langle \bar{c}_n : n < \omega \rangle$ is an indiscernible sequence over B , $p \vdash \psi(\bar{x}, \bar{c}_0)$ and $\{\psi(\bar{x}, \bar{c}_n) : n < \omega\}$ is contradictory as T is dependent.

- 3) By the definitions (or see [Sh:93]).
- 4) Easy or see [Sh 3] or [Sh:93].
- 5) Easy.
- 6) By part (2) and transitivity of “equality of types” and Fact 5.5.
- 7), 8), 9) Easy.
- 10) See 5.11. □_{5.4}

We implicitly use the easy

5.5 Fact. 1) If I is a linear order, \bar{s}_0, \bar{s}_1 are increasing n -tuples from I then we can find an increasing n -tuple \bar{s}_2 from I such that

- ⊗_ω there is a linear order $J \supseteq I$ such that for $\ell \in \{0, 1\}$ there is an indiscernible sequence $\langle \bar{t}_k : k < \omega \rangle$ of increasing n -tuples from J such that $\{\bar{s}_\ell, \bar{s}_2\} = \{\bar{t}_0, \bar{t}_1\}$; indiscernible means for quantifier free formulas in J .

- 2) Similar for $\langle \bar{b}_t : t \in I \rangle$ an indiscernible sequence over A in \mathfrak{C} .

Proof. 1) Let $J \supseteq I$ be dense with no last elements. Choose for $k = 1, 2, \dots$ an increasing sequence \bar{t}_k of length n from J such that $\text{Rang}(\bar{s}_0 \hat{\ } \bar{s}_1) < \text{Rang}(\bar{t}_k) < \text{Rang}(\bar{t}_{k+1})$. So $\langle \bar{s}_\ell \rangle \hat{\ } \langle \bar{t}_1, \bar{t}_2, \dots \rangle$ is an indiscernible sequence in J for $\ell = 0, 1$.

- 2) Easy. □_{5.5}

5.6 Observation. Assume $C_1 \subseteq C_2$ and $B \subseteq A$ and $\text{tp}(C_1, A)$ does not fork over B . Then for some elementary mapping f , $\text{Dom}(f) = A \cup C_2$, $f \upharpoonright A = \text{id}_A$, $f \upharpoonright C_1 = \text{id}_{C_1}$ and $\text{tp}(f(C_2), A)$ does not fork over B .

Proof. By observation 5.2(3).

5.7 Definition. 1) Let p be an m -type, $p \upharpoonright B_2 \in \mathcal{S}^m(B_2)$. We say that p strictly does not divide over (B_1, B_2) , (when $B_1 = B_2 = B$ we may write “over B ”) if:

- (a) p does not divide over B_1
- (b) $B_1, B_2 \subseteq \mathfrak{C}$
- (c) if $\langle \bar{c}_n : n < \omega \rangle$ is an indiscernible sequence over B_2 such that \bar{c}_0 realizes p and A is any set satisfying $\text{Dom}(p) \subseteq A$, then there is an indiscernible sequence $\langle \bar{c}'_n : n < \omega \rangle$ over A such that \bar{c}'_0 realizes p and $\text{tp}(\langle \bar{c}_n : n < \omega \rangle, B_2) = \text{tp}(\langle \bar{c}'_n : n < \omega \rangle, B_2)$ (instead “ \bar{c}_0 realizes p ”) (equivalently we can demand \bar{c}_0 realize $p \upharpoonright B_2$).

1A) “Strictly divide” is the negation.

2) We say that p strictly forks over (B_1, B_2) iff $p \vdash \bigvee_{\ell < n} \varphi_\ell$ for some $\langle \varphi_\ell : \ell < n \rangle$ such that $(p \upharpoonright B_2) \cup \{\varphi_\ell\}$ strictly divides over (B_1, B_2) for each $\ell < n$.

We shall need some parallel for strictly does not fork.

5.8 Observation. 1) “Strictly does not divide/fork over p ” is perserved by permuting the variables, repeating variables and by automorphisms of \mathfrak{C} and if it holds for $\text{tp}(\bar{b} \hat{\ } \bar{c}, A)$ then it holds for $\text{tp}(\bar{b}, A)$.

2) If p strictly does not fork over (B_1, B_2) then p strictly does not divide over (B_1, B_2) .

3) If p strictly does not divide over B then p does not divide over B .

4) If p strictly does not fork over B then p does not fork over B .

5) If p is an m -type which strictly does not fork over (B_1, B_2) and $\text{Dom}(p) \subseteq A$ then some $q \in \mathcal{S}^m(A)$ extending p strictly does not fork over (B_1, B_2) .

6) If $B_1 \subseteq B'_1, B'_2 \subseteq B_2$ and $p(\bar{x}) \vdash p'(\bar{x})$ and $p(\bar{x})$ strictly does not divide/fork over (B_1, B_2) and $p' \upharpoonright B_2$ is complete then $p'(\bar{x})$ strictly does not divide/fork over (B'_1, B'_2) .

7) In Definition 5.7, clause (c) the case $A = \text{Dom}(p) \cup B_2$ suffice.

8) If p strictly forks over (B_1, B_2) then for some finite $q \subseteq p$ the type $q \cup (p \upharpoonright B_2)$ strictly forks over (B_1, B_2) .

9) If p is finitely satisfiable in M , then p strictly does not fork over M .

10) [?] If p does not fork over B_1 and $B'_2 \subseteq \mathfrak{C}$ then for some B_2 of cardinality $\leq |B'_2| + |T|$, then type p strictly does not fork over (B_1, B_2) .

Proof. Easy, e.g.,

5) Use “rich enough” A .

6) Recall in Definition 5.7, clause (c) we can demand only “ \bar{c}_0 realizes $p \upharpoonright B_2$ ” and for any such $\langle \bar{c}_n : n < \omega \rangle$ there is \bar{c}_0'' realizing p hence \bar{c}_0, \bar{c}_0'' realizes the same type over B_2 hence there automorphism F of \mathfrak{C} over B_2 mapping \bar{c}_0 to \bar{c}_0'' and use the definition for $\langle F(\bar{c}_n) : n < \omega \rangle$.

The next claim is a parallel of: every type over A does not fork over some “small” $B \subseteq A$. If we have “ p is over A implies p does not fork over A ” we could have improvement.

More elaborately, note that if M is a dense linear order $p \in \mathcal{S}(M)$, then p actually corresponds to a Dedekind cut of M . So though in general p is not definable, $p \upharpoonright \{x \in M : x \notin (a, b)\}$ is definable whenever (a, b) is an interval of M which includes the cut. So p is definable in large pieces. The following (as well as 5.18) realizes the hope that something in this direction holds for every dependent theory.

5.9 Claim. *If $p \in \mathcal{S}^m(A)$ and $B \subseteq A$, then we can find $C \subseteq A$ of cardinality $\leq |T|$ such that:*

- ⊗ *if $\langle \bar{a}_n : n < \omega \rangle$ is an indiscernible sequence over $B \cup C$ such that $\bar{a}_0 \subseteq A$ and $\text{tp}(\bar{a}_0, B \cup C)$ does not fork over B and $\{\varphi(\bar{x}, \bar{a}_n) : n < \omega\}$ is contradictory or just $\varphi(\bar{x}, \bar{a}_0)$ forks over $B \cup C$, then $\neg\varphi(x, \bar{a}_0) \in p$ (in other words no formula in p divides over $B \cup C$ when the type of the sequence of parameters over $B \cup C$ does not fork over B).*

5.10 Conclusion. 1) For every $p \in \mathcal{S}^m(A)$ and $B \subseteq A$, we can find $C \subseteq A$, $|C| \leq |T|$ such that:

- ⊗ if $\langle \bar{a}_n : n < \omega \rangle$ is an indiscernible sequence over $B \cup C$ satisfying $\bar{a}_0 \cup \bar{a}_1 \subseteq A$ and $\text{tp}(\bar{a}_0 \hat{\ } \bar{a}_1, B \cup C)$ does not fork over B , then for any φ
 - (*) $\varphi(x, \bar{a}_0) \in p$ iff $\varphi(x, \bar{a}_1) \in p$.

2) For every $\bar{x} = \langle x_\ell : \ell < m \rangle$ and formula $\varphi = \varphi(\bar{x}; \bar{y})$ for some finite $\Delta \subseteq \mathbb{L}(T)$ we have:

if $p \in \mathcal{S}^m(A)$, $B \subseteq A$, then for some finite $C \subseteq A$ (even with a bound depending on m, φ, T only), we have:

if $\langle \bar{a}_\ell : \ell < k \rangle$ is Δ -indiscernible sequence over $B \cup C$ and $\text{tp}_\Delta(\bar{a}_0 \hat{\ } \bar{a}_1, B \cup C)$ does not fork over A , then $\varphi(x, \bar{a}_0) \in p \Leftrightarrow \varphi(x, \bar{a}_1) \in p$.

3) The local version of 5.9 holds with a priori finite bound on B .

Remark. If $A = |M|$, then we can replace strong splitting by dividing by 5.4(10).

Proof of 5.9. By induction on $\alpha < |T|^+$ we try to choose $C_\alpha, \bar{a}_\alpha, k_\alpha$ and $\langle \bar{a}_{\alpha,n}^k : n < \omega \rangle$ and $\varphi_\alpha(\bar{x}, \bar{y}_\alpha), \varphi_{\alpha,k}(\bar{x}, \bar{y}_{\alpha,k})$ such that:

- (a) $C_\alpha = \cup\{\bar{a}_\beta : \beta < \alpha\} \cup B$
- (b) $\langle \bar{a}_{\alpha,n}^k : n < \omega \rangle$ is an indiscernible sequence over C_α for $k < k_\alpha$
- (c) $\bar{a}_\alpha \subseteq A$
- (d) $\varphi_\alpha(\bar{x}, \bar{a}_\alpha) \in p$
- (e) $\{\varphi_{\alpha,k}(\bar{x}, \bar{a}_{\alpha,n}^k) : n < \omega\}$ is contradictory
- (f) $\text{tp}(\bar{a}_\alpha, B \cup C_\alpha)$ does not fork over B
- (g) $\varphi_\alpha(\bar{x}, \bar{a}_\alpha) \vdash \bigvee_{k < k_\alpha} \varphi_{\alpha,k}(\bar{x}, \bar{a}_{\alpha,0}^k)$.

If for some $\alpha < |T|^+$ we are stuck, $C = C_\alpha \setminus B$ is as required. So assume that we have carried the induction and we shall eventually get a contradiction.

By induction on $\alpha < |T|^+$ we choose $D_\alpha, F_\alpha, \langle \bar{b}_{\beta,n}^k : n < \omega \rangle$ for $\beta < \alpha$ such that (but $\bar{b}_{\alpha,n}^k$ are defined in the $(\alpha + 1)$ -th stage):

- (α) F_α is an elementary mapping, increasing continuous with α
- (β) $\text{Dom}(F_\alpha) = C_\alpha, \text{Rang}(F_\alpha) \subseteq D_\alpha$
- (γ) $D_\alpha = \text{Rang}(F_\alpha) \cup \{\bar{b}_{\beta,n}^k : \beta < \alpha, k < k_\alpha \text{ and } n < \omega\}$ so $D_\alpha \subseteq \mathfrak{C}$ is increasing continuous
- (δ) $\langle \bar{b}_{\alpha,n}^k : n < \omega \rangle$ is an indiscernible sequence over $F_\alpha(C_\alpha)$
- (ε) $F_{\alpha+1}(\bar{a}_\alpha) = \bar{b}_\alpha$
- (ζ) Some automorphism $F_{\alpha+1}^+ \supseteq F_{\alpha+1}$ of \mathfrak{C} maps $\bar{a}_{\alpha,n}^k$ to $\bar{b}_{\alpha,n}^k$ for $n < \omega, k < k_\alpha$.

For $\alpha = 0$, α limit this is trivial. For $\alpha = \beta + 1$, clearly $F_\alpha(\text{tp}(\bar{a}_\alpha, C_\alpha))$ is a type in $\mathcal{S}^{<\omega}(F(C_\alpha))$ which does not fork over $F_\alpha(B) = F_0(B)$ hence has an extension $q_\alpha \in \mathcal{S}^{<\omega}(D_\alpha)$ which does not fork over $F_0(B)$ and let \bar{b}_α realize it. Let $F_{\alpha+1} \supseteq F_\alpha$ be the elementary mapping extending F_α with domain $C_{\alpha+1}$ mapping \bar{a}_α to \bar{b}_α . Let $F_{\alpha+1}^+ \supseteq F_\alpha$ be an automorphism of \mathfrak{C} and let $\bar{b}_{\alpha,n}^k = F_{\alpha+1}^+(\bar{a}_{\alpha,n}^k)$ for $n < \omega, k < k_\alpha$. So $D_{\alpha+1}, F_{\alpha+1}$ are defined.

Having carried the induction let $F \supseteq \cup\{F_\alpha : \alpha < |T|^+\}$ be an automorphism of \mathfrak{C} . We claim that for each $\alpha < |T|^+$ and $k < k_\alpha$, for every $\beta \in (\alpha, |T|^+]$ we have

$$(*)_\beta \quad \langle \bar{b}_{\alpha,n}^k : n < \omega \rangle \text{ is an indiscernible sequence over } F_\alpha(C_\alpha) \cup \{\bar{b}_\gamma : \gamma \in [\alpha+1, \beta)\}.$$

We prove this by induction on β . For $\beta = \alpha$ this holds by clause (δ), for $\beta \equiv \alpha + 1$ this is the same as for $\beta = \alpha$. For β limit use the definition of indiscernibility. For $\beta = \zeta + 1$ use $\text{tp}(\bar{b}_\zeta, D_\gamma)$ does not fork over $F(B)$ hence over $F_\alpha(C_\alpha) \cup \{F_{\gamma+1}(\bar{b}_\gamma) : \alpha < \gamma < \zeta\}$ by 5.4(5); so by the induction hypothesis and 5.4(6) clearly $(*)_\beta$ holds.

Now without loss of generality F_α is the identity.

From $(*)_{|T|^+}$ we can conclude

$$(**) \quad \text{for any } n < \omega \text{ and } \alpha_0 < \dots < \alpha_{n-1} < |T|^+ \text{ and } \nu \in \prod_{\ell < n} k_{\alpha_\ell} \text{ and } \eta \in {}^n 2 \text{ the sequences } \bar{a}_{\alpha_0,0}^{\nu(0)} \wedge \bar{a}_{\alpha_1,0}^{\nu(1)} \wedge \dots \wedge \bar{a}_{\alpha_{n-1},0}^{\nu(n-1)} \text{ and } \bar{a}_{\alpha_0,\eta(0)}^{\nu(0)} \wedge \bar{a}_{\alpha_1,\eta(1)}^{\nu(1)} \wedge \dots \wedge \bar{a}_{\alpha_{n-1},\eta(n-1)}^{\nu(n-1)} \text{ realizes the same type over } B.$$

[Why? By induction on $\max\{\ell : \eta(\ell) = 1 \text{ or } \ell = -1\}$.]

Let \bar{c} realize $F(p)$.

Now as $\{\varphi_{\alpha,k}(x, \bar{b}_{\alpha,n}) : n < \omega\}$ is contradictory there is $n = n[\alpha] < \omega$ such that $\mathfrak{C} \models \neg \varphi_{\alpha,k}(\bar{c}, \bar{b}_{\alpha,n})$, whereas $\mathfrak{C} \models \varphi[\bar{c}, \bar{b}_{\alpha,0}]$ as $\varphi(\bar{x}, \bar{b}_{\alpha,0}) \in F(\alpha)$; by renaming without loss of generality $\models \neg \varphi_{\alpha,k}[c, \bar{b}_{\alpha,n}^k]$ for $\alpha < |T|^+, n \in [1, \omega)$. Now if $n <$

$\omega, \alpha_0 < \dots < \alpha_{n-1} < |T|^+$ and $\eta \in {}^n 2$ then $\mathfrak{C} \models \bigwedge_{\ell < m} \varphi_{\alpha_\ell, k}(\bar{c}, \bar{b}_{\alpha_\ell, \eta(\ell)})^{\eta(\ell)}$ hence $\mathfrak{C} \models (\exists \bar{x})[\bigwedge_{\ell < n} \varphi_{\alpha_\ell, k}(\bar{x}, \bar{b}_{\alpha_\ell, \eta(\ell)})^{\eta(\ell)}]$ hence by $(**)$ we have $\mathfrak{C} \models (\exists \bar{x})[\bigwedge_{\ell < n} \varphi_{\alpha_\ell, k}(\bar{x}, \bar{b}_{\alpha_\ell, 0})^{\eta(\ell)}]$.

Hence the independence property holds, contradiction. $\square_{5.9}$

Proof of 5.10. 1) Follows from 5.9 by 5.4(2).

2) By 5.9 and compactness or repeating the proof.

3) Assume that $A_0 \subseteq A_1 \cap A_2$ over A_1 , it forks over A_0 and $\text{tp}(A_1/A_2)$ does not fork over A_0 . Then p forks over A_2 . $\square_{5.10}$

5.11 Claim. 1) Assume p a type, $B \subseteq M$, $\text{Dom}(p) \subseteq M$ and M is $|B|^+$ -saturated. Then

- (A) p does not fork over B iff p has a complete extension over M which does not fork over B iff p has a complete extension over M which does not divide over B iff p has a complete extension over M which does not split strongly over B
- (B) if $p = \text{tp}(\bar{c}, M)$ and $\varphi(x, \bar{a}) \in p$ forks over B , then for some $\langle \bar{a}_n : n < \omega \rangle$ indiscernible over B , $\{\bar{a}_n : n < \omega\} \subseteq M$, $\bar{a}_0 = a$ and $\neg\varphi(\bar{x}, \bar{a}_1) \in p$, and of course, $\varphi(\bar{x}, \bar{a}_0) \in p$.

2) Assume $\text{tp}(C_1/A)$ does not fork over $B \subseteq A$ and $\text{tp}(C_2, (A \cup C_1))$ does not fork over $B \cup C_1$. Then $\text{tp}(C_1 \cup C_2, A)$ does not fork over B .

Proof. 1) Read the definitions.

Clause (A):

First implies second by 5.4(3), second implies third by Definition 5.2, third implies fourth by 5.4(2). If the first fails, then $p \vdash \bigvee_{\ell < k} \varphi_\ell(\bar{x}, \bar{a}_\ell)$ for some k where each

$\varphi_\ell(\bar{x}, \bar{a}_\ell)$ divides over B ; let $\langle \bar{a}_{\ell, n} : n < \omega \rangle$ witness this hence $\bar{a}_\ell = \bar{a}_{\ell, 0}$. As M is $|B|^+$ -saturated, without loss of generality $\bar{a}_{\ell, n} \subseteq M$. So for every $q \in \mathcal{S}^m(M)$ extending p , for some $\ell < k$, $\varphi_\ell(\bar{x}, \bar{a}_\ell) \in q$ but for every large enough n , $\neg\varphi_\ell(\bar{x}, \bar{a}_{\ell, n}) \in q$, so q splits strongly, i.e., fourth fails. So fourth implies first.

Clause (B):

Similar.

2) Let M be $|B|^+$ -saturated model such that $A \subseteq M$. By 5.4(3) there is an elementary mapping f_1 such that $f_1 \upharpoonright B = \text{id}_B$ and $\text{Dom}(f_1) = C_1 \cup A$ and $f_1(C_1)/M$ does not fork over B . Similarly we can find an elementary mapping

$f \supseteq f_1$ such that $\text{Dom}(f) = C_1 \cup C_2 \cup A$ and $f(C_2)/(M \cup f(C_1))$ does not fork over $A \cup f(C_1)$. By 5.4(2), $f_1(C_1)/M$ does not split strongly over B . Again by 5.4(2), $f(C_2)/(M \cup f_1(C_1))$ does not split strongly over $B \cup f_1(C_1)$. Together if $\bar{\mathbf{b}} \subseteq M$ is an infinite indiscernible sequence over B then it is an indiscernible sequence over $f(C_1) \cup B$ and even over $f(C_2) \cup (f(C_1) \cup B)$ (use the two previous sentences and 5.4(6)). But this means that $f(C_1) \cup f(C_2)/M$ does not split strongly over B , (here the exact version of strong splitting we choose is immaterial as M is $|B|^+$ -saturated). So by 5.11(1) we get that $f(C_1) \cup f(C_2)/M$ does not fork over B hence $f(C_1 \cup C_2)/A$ does not fork over B but $f \supseteq \text{id}_A$ so also $C_1 \cup C_2/A$ does not split over B . $\square_{5.11}$

5.12 Definition. 1) We say $\langle \bar{a}_t : t \in J \rangle$ is a nonforking sequence over (B, A) if $B \subseteq A$ and for every $t \in J$ we have $\text{tp}(\bar{a}_t, A \cup \cup \{\bar{a}_s : s <_J t\})$ does not fork over B . 2) We say that $\langle \bar{a}_t : t \in J \rangle$ is a strict nonforking sequence over (B_1, B_2, A) if $B_1 \subseteq B_2 \subseteq A$ and for every $t \in J$ the type $\text{tp}(\bar{a}_t, A \cup \cup \{\bar{a}_s : s <_J t\})$ strictly does not fork over (B_1, B_2) , see Definition 5.7. 3) We say $\mathcal{A} = (A, \langle (\bar{a}_\alpha, B_\alpha) : \alpha < \alpha^* \rangle)$ is an \mathbf{F}_κ^f -construction or $\langle (a_\alpha, B_\alpha) : \alpha < \alpha^* \rangle$ an \mathbf{F}_κ^f -construction over A if $B_\alpha \subseteq A_\alpha = A \cup \cup \{\bar{a}_\beta : \beta < \alpha\}$ has cardinality $< \kappa$ and $\text{tp}(\bar{a}_\alpha, A_\alpha)$ does not fork over B_α .

5.13 Claim. 1) Assume

- (a) $\langle \bar{a}_t : t \in J \rangle$ is a nonforking sequence over (B, A)
- (b) $\langle \bar{b}_{t,n} : n < \omega \rangle$ is an indiscernible sequence over A , each $\bar{b}_{t,n}$ realizing $\text{tp}(\bar{a}_t, A)$ for each $t \in J$.

Then we can find $\bar{a}_{t,n}$ for $t \in J, n < \omega$ such that:

- (α) $\langle \bar{a}_{t,n} : n < \omega \rangle$ is an indiscernible sequence over $A \cup \cup \{\bar{a}_{s,0} : s \neq t\}$
- (β) $\text{tp}(\langle \bar{b}_{t,n} : n < \omega \rangle, A) = \text{tp}(\langle \bar{a}_{t,n} : n < \omega \rangle, A)$
- (γ) $\bar{a}_{t,0} = \bar{a}_t$.

2) Assume

- (a) $\langle \bar{a}_t : t \in J \rangle$ is a strictly nonforking sequence over (B_j, B, A)
- (b) $\langle \bar{b}_{t,n} : n < \omega \rangle$ is an indiscernible sequence over B each $\bar{b}_{t,n}$ realizing $\text{tp}(\bar{a}_t, B)$.

Then we can find $\bar{a}_{t,n}$ for $t \in J, n < \omega$ such that

- (α) $\langle \bar{a}_{t,n} : n < \omega \rangle$ is an indiscernible sequence over $A \cup \{\bar{a}_{s,n} : n < \omega, s \in J \setminus \{t\}\}$
- (β) $\text{tp}(\langle \bar{b}_{t,n} : n < \omega \rangle, B) = \text{tp}(\langle \bar{a}_{t,n} : n < \omega \rangle, B)$
- (γ) $\bar{a}_{t,0} = \bar{a}_t$.

Proof. We prove by induction on $|J|$, for all cases.

Case 1: J is finite. For part (1) this holds by the proof of 5.9. We now deal with part (2).

We prove this by induction on n , for $n = 0, 1$ this is trivial; assume we have proved for n and we shall prove for $n + 1$. Let $\lambda = (|A| + |T|)^+$.

So let $J = \{t_\ell : \ell \leq n\}$. First we can find an indiscernible sequence $\langle \bar{a}_{t_0, \alpha} : \alpha < \lambda \rangle$ over A such that $\bar{a}_{t_0, 0} = \bar{a}_{t_0}$ and for some automorphism F of \mathfrak{C} over A we have $n < \omega \Rightarrow F(\bar{b}_{t_0, n}) = \bar{a}_{t_0, n}$. Let $A' =: A \cup \{\bar{a}_{t_0, \alpha} : \alpha < \lambda\}$. This is possible by Definition 5.7.

Second, we can choose \bar{a}'_{t_ℓ} by induction on ℓ such that $\bar{a}'_{t_0} = \bar{a}_{t_0}$ and if $\ell > 0$ then $\text{tp}(\bar{a}'_{t_\ell}, A' \cup \cup\{\bar{a}'_{t_m} : m = 1, \dots, \ell - 1\})$ strictly does not fork over B and the two sequences $\bar{a}_{t_0} \hat{\ } \dots \hat{\ } \bar{a}_{t_\ell}, \bar{a}'_{t_0} \hat{\ } \dots \hat{\ } \bar{a}'_{t_\ell}$ realizes the same type over A . We can do it by 5.2(5) and nonforking being preserved by elementary mapping. By 5.11(2) the type $\text{tp}(\bar{a}'_{t_1} \hat{\ } \dots \hat{\ } \bar{a}'_{t_n}, A')$ does not fork over B hence by 5.4(6) the sequence $\langle \bar{a}_{t_0, \alpha} : \alpha < \lambda \rangle$ is an indiscernible sequence over $A \cup (\bar{a}'_{t_1} \hat{\ } \dots \hat{\ } \bar{a}'_{t_n})$.

Now we use the induction hypothesis with $B, A', \langle \bar{a}'_{t_\ell} : \ell = 1, \dots, n \rangle, \langle b_{t_\ell, m} : m < \omega \rangle$ for $\ell = 1, \dots, n$ and let $\langle \bar{a}'_{t_\ell, n} : n < \omega \rangle$ for $\ell = 1, \dots, n$ be as in the claim.

By [Sh 715] for some $\alpha^* < \lambda$ the sequence $\langle \bar{a}'_{t_0, \alpha} : \alpha \in [\alpha^*, \alpha^* + \omega) \rangle$ is an indiscernible sequence over $A \cup \cup\{\bar{a}'_{t_\ell, m} : m < \omega, \ell = 1, \dots, n\}$ and as $A' = A \cup \{\bar{a}'_{t_0, \alpha} : \alpha < \lambda\}$ clearly for $\ell = 1, \dots, n$ the sequence $\langle \bar{a}'_{t_\ell, m} : m < \omega \rangle$ is indiscernible over $A \cup \cup\{\bar{a}'_{t_k, m} : k \in \{1, \dots, n\} \setminus \{\ell\}\} \cup \cup\{\bar{a}'_{\alpha^* + m} : m < \omega\}$. But we know that $\langle \bar{a}'_{t_0, \alpha} : \alpha < \alpha^* + \omega \rangle$ is an indiscernible sequence over $A \cup \{\bar{a}'_{t_\ell} : \ell = 1, \dots, n\}$, hence the sequence $\bar{a}'_{t_0, \alpha^*} \hat{\ } \bar{a}'_{t_1} \hat{\ } \dots \hat{\ } \bar{a}'_{t_n}$ realizes over A the same type as $\bar{a}'_{t_0, 0} \hat{\ } \bar{a}'_{t_1} \hat{\ } \dots \hat{\ } \bar{a}'_{t_n}$ hence as $\bar{a}_{t_0} \hat{\ } \bar{a}_{t_1} \hat{\ } \dots \hat{\ } \bar{a}_{t_n}$. So for some automorphism F of \mathfrak{C} , $F \upharpoonright A = \text{id}_A$, $\ell = 1, \dots, n \Rightarrow \bar{a}_{t_\ell} = F(\bar{a}'_{t_\ell, 0})$ and $\bar{a}_{t_0} = F(\bar{a}'_{t_0, \alpha^*})$ and let $\bar{a}_{t_\ell, m} = F(\bar{a}'_{t_\ell, m})$ for $\ell = 1, \dots, n$ and $m < \omega$ and $\bar{a}_{t_0, m} = F(\bar{a}'_{t_0, \alpha^* + m})$.

So we are done.

Case 2: J infinite.

By Case 1 + compactness. □_{5.13}

Remark. Can we use just no dividing?

5.14 Claim. 1) Assume $\langle A_t : t \in J \rangle$ is a nonforking sequence over (B, A) and $C_t \subseteq \mathfrak{C}$ for $t \in J$. Then we can find $\langle f_t : t \in J \rangle$ such that

(a) f_t is an elementary mapping with domain

$$A \cup A_t \cup C_t$$

(b) $f_t \upharpoonright (A \cup A_t)$ is the identity

(c) $\text{tp}(A_t, A \cup \cup\{A_s \cup f_s(C_s) : s < t\})$ does not fork over B .

2) If in addition $\text{tp}(C_t, A \cup A_t)$ does not fork over $A \cup A_t$ then we can add

(c)⁺ $\langle A_t \cup f_t(C_t) : t \in J \rangle$ is a nonforking sequence over (B, A) .

Remark. 1) What about \mathbf{F}^f -construction? Non well-ordered construction (i.e., the indiscernible set is)?

2) In 5.14(2) we may weaken the assumption to: for every $A' \supseteq A$ the $A_t \cup C_t/A$ can be embedded to a complete nonforking type over A' .

Proof. 1) As in the proof of 5.9.

2) Similarly.

5.15 Claim. 1) Assume

(a) $\langle A_t : t \in J \rangle$ is a nonforking sequence over (B, A) .

Then for any initial segment I of J , $\text{tp}(\cup\{A_t : t \in J \setminus I\}, A \cup \{A_t : t \in J\})$ does not fork over B .

2) Assume (a) and

(b) $\langle \bar{a}_{t,n} : n < \omega \rangle$ is an indiscernible sequence over A ,

(c) $\bar{a}_{t,n} \in {}^\omega(A_t)$

(d) $\langle \bar{a}_{t,n} : n < \omega \rangle$ is an indiscernible sequence over $A \cup \cup\{A_s : s <_J t\}$.

Then $\langle \langle \bar{a}_{t,n} : n < \omega \rangle : t \in J \rangle$ are mutually indiscernible over A . Also for any nonzero $k < \omega$ and $t_0 < \dots < t_{k-1}$ in J the sequences $\langle \bar{a}_{t_\ell,n} : n < \omega \rangle$ for all $\ell < k$ are mutually indiscernible over $A \cup \cup\{A_s : \neg(t_0 \leq s \leq t_{k-1})\}$.

5.16 Question: If $n_\ell < \omega$ for $\ell < n$ does the sequences $\langle \bar{a}_{t_0,n_0} \hat{\ } \bar{a}_{t_1,n_1} \hat{\ } \dots \hat{\ } \bar{a}_{t_{k-1},n_{k-1}} \rangle$ and $\bar{a}_{t_0,0} \hat{\ } \bar{a}_{t_1,0} \hat{\ } \dots \hat{\ } \bar{a}_{t_{k-1},0}$ realize the same type over $A \cup \cup\{A_s : s <_J t_0 \text{ and } s_J > t_{k-1}\}$. Need less?

Proof. 1) for $J \setminus I$ finite, by induction on $|J \setminus I|$ using 5.11(2). The general case follows by 5.4(7) [similarly for \mathbf{F}_κ^f -construction].

2) For $k = 1$ this follows by 5.4(6) + 5.11(2) using part (1) with $A \cup \cup\{A_s : s < t\}$, $\langle A_r : r \in J, r_J \geq t \rangle$ instead A , $\langle A_r : r \in J \rangle$.

For $k+1 > 1$, given $t_0 <_J \dots <_J t_k$ and $n_0 < \omega, \dots, n_k < \omega$. Use the case $k = 1$ for each t_ℓ and combine. $\square_{5.15}$

Remark. 1) If $p \in \mathcal{S}^m(M)$, M is quite saturated, divide = fork.

2) Try on existence of indiscernibility. (omit?)

5.17 Claim. Assume that $\langle \bar{a}_t : t \in J \rangle$ is a nonforking sequence over (B, A) and $A = |M|$.

- 1) For every (finite sequence) \bar{b} the set $\{t : \bar{b}/(A \cup \bar{a}_t) \text{ forks over } \bigcup_{s < t} a_s \cup A \text{ or if } A \text{ is a model over } A\}$ has cardinality $\leq |T|$. (Saharon: consider dividing).
 2) For each $\varphi(\bar{x}, \bar{y}, \bar{z})$ for some $n = n_{\varphi(\bar{x}, \bar{y})}$ the set $W_{\bar{b}}^\varphi = \{t : \text{tp}_\varphi(\bar{b}, A \cup \bar{a}_t) \text{ forks over } A\}$ has $\leq n$ members.

Proof. 1) By (2).

2) Assume toward contradiction that this fails for n , so without loss of generality, $J = \{t_0, \dots, t_{n-1}\} = W_{\bar{b}}^\varphi$. Choose λ_ℓ, M_ℓ such that $\lambda_0 = |A| + |T|$, $M_\ell = 2^{\lambda_\ell}$, $\lambda_{\ell+1} = M_{\ell+1}$. We can find $\langle M_\ell : \ell \leq n \rangle$ such that M_ℓ is λ_ℓ^+ -saturated, $\|M_\ell\| = M_\ell$ and $A \cup \{\bar{a}_{t_k} : k < \ell\} \subseteq M_\ell \prec M_{\ell+1}$ and $\text{tp}(\bar{a}_{t_\ell}, M_\ell)$ does not fork over B . By 5.4(C), $\text{tp}(\bar{b}, A \cup \bar{a}_{t_\ell})$ forks over A .

5.18 Claim. Assume that $p(\bar{x})$ is a type of cardinality $< \kappa$ which does not fork over A . Then for some $B \subseteq A$ of cardinality $< \kappa + |T|$, the type p does not fork over B .

Proof. Without loss of generality p is closed under conjunction.

For any finite sequence $\bar{\varphi} = \langle (\varphi_\ell(\bar{x}, \bar{y}_\ell), n_\ell) : \ell < n \rangle$ and formula $\psi(x, \bar{c}) \in p$ and set $B \subseteq A$ we define

$$\begin{aligned} \Gamma_{\Gamma, \bar{\varphi}, \psi(\bar{x}, \bar{c})} &= \{(\forall x)(\psi(x, \bar{c}) \rightarrow \bigvee_{\ell < n} \varphi_\ell(\bar{x}, \bar{y}_{\ell, 0}))\} \cup \\ &\quad \{\neg(\exists \bar{x}) \bigwedge_{n \in w} \varphi_\ell(\bar{x}, y_{\ell, n}) : \ell < n \text{ and } w \in [\omega]^{n_\ell}\} \cup \\ &\quad \{\vartheta(y_{\ell, m_1}, \dots, y_{\ell, m_k}, \bar{b}) = \vartheta(y_{\ell, 0}, \dots, y_{\ell, k}, \bar{b}) : \\ &\quad \bar{b} \subseteq B, \vartheta \in \mathbb{L}(T), m_1 < \dots < m_k < \omega\}. \end{aligned}$$

Now as p does not fork over A clearly for any $\bar{\varphi}$ as above and $\psi(\bar{x}, \bar{c}) \in p$ the set $\Gamma_{A, \bar{\varphi}, \psi(\bar{x}, \bar{c})}$ is inconsistent hence for some finite set $B = B_{\bar{\varphi}, \psi(\bar{x}, \bar{c})} \subseteq A$ the set $\Gamma_{B, \bar{\varphi}, \psi(\bar{x}, \bar{c})}$ of formulas is inconsistent. Now $B^* = \cup\{B_{\bar{\varphi}, \psi(\bar{x}, \bar{c})} : \psi(\bar{x}, \bar{c}) \in p \text{ and } \bar{\varphi} \text{ is as above}\}$ is as required. $\square_{5.18}$

The following is another substitute for “every type p does not fork over a small subset of $\text{Dom}(p)$ ”; this is related to 5.9(3).

5.19 Claim. Assume $p \in \mathcal{S}^m(M)$ and $B \subseteq M$. Then we can find C such that

- $(*)_1$ $C \subseteq M$ and $|C| \leq |T| + |B|$ and
- $(*)_M, B, C^p$ if $D \subseteq M$ and $\text{tp}(D/B \cup C)$ does not fork over B then $p \restriction (B \cup D)$ does not fork over $B \cup C$.

Proof. Follows by 5.9.

* * *

5.20 Claim. Assume that T is strongly dependent (see 3.2)

- (1) If $p \in \mathcal{S}(A)$ and $B \subseteq A$ is finite, then for some finite C we have $B \subseteq C \subseteq A$ and
 - $(*)$ if $\bar{a} \subseteq A$, $\bar{a} \bigcup_B C$ then $p \restriction (B + \bar{a})$ does not fork over B .
- (2) If $\bigcup_B \{C_n : n < \omega\}$ and $B \cup \bigcup \{C_n : n < \omega\} \subseteq A$, then for no $p \in \mathcal{S}^m(A)$ do we have $p \restriction (B \cup C_n)$ forks over B for every $n < \omega$ [can be chosen as the definition].

Proof. Should be clear.

5.21 Definition. We say $\{\bar{a}_\alpha : \alpha < \alpha^*\}$ is ℓ -independent over A if: we can find $\bar{a}_{\alpha,n}$ (for $\alpha < \alpha^*$, $n < \omega$) such that:

- (a) $\bar{a}_\alpha = \bar{a}_{\alpha,0}$
- (b) $\langle \bar{a}_{\alpha,n} : n < \omega \rangle$ is an indiscernible sequence over $A \cup \{a_{\beta,m} : \beta \in \alpha^* \setminus \{\alpha\} \text{ and } m < \omega\}$
 - (α) if $\ell = 1$ then for some $\bar{b}_n \in A$ ($n < \omega$) for every $\alpha < \alpha^*$ we have $\langle \bar{b}_n : n < \omega \rangle^\wedge \langle \bar{a}_{\alpha,n} : n < \omega \rangle$ is an indiscernible sequence
 - (β) if $\ell = 2$ then for some $\bar{b}_{\alpha,n} \subseteq A$ (for $\alpha < \alpha^*$, $n < \omega$), $\langle \bar{b}_{\alpha,n} : n < \omega \rangle^\wedge \langle \bar{a}_{\alpha,n} : n < \omega \rangle$ is an indiscernible sequence.

We now show that even a very weak version of independence has limitations.

5.22 Claim. 1) For every finite $\Delta \subseteq \mathbb{L}(\tau_T)$ there is $n^* < \omega$ such that we cannot find $\bar{\varphi} = \langle \varphi_n(\bar{x}, \bar{a}_n) : n < n^* \rangle$ such that

$(*)_{\bar{\varphi}}$ for each n there are $m_n < \omega$ and $\langle \bar{b}_{m,\ell}^n : \ell < \omega, m < m_n \rangle$ and $\langle \psi_m^n(\bar{x}, \bar{y}_n) : m < m_n \rangle$ such that

(α) $\langle \bar{b}_{m,\ell}^n : \ell < \omega \rangle$ is an indiscernible sequence over $\cup \{\bar{a}_k : k < n^*, k \neq n\}$

(β) $\bar{b}_{m,0}^n = \bar{a}_n$

(γ) $\{\psi_m^n(x, \bar{b}_{m,\ell}^n : \ell < \omega)\}$ is contradictory

(δ) $\psi_m^n(\bar{x}, \bar{y}_n) \in \Delta$

(ε) $\varphi_n(\bar{x}, \bar{a}_n) \vdash \bigvee_{m < m_n} \psi_m^n(\bar{x}, \bar{a}_n)$

(ζ) $\models (\exists \bar{x}) \bigwedge_{n < n^*} \varphi_n(\bar{x}, \bar{a}_n)$.

2) We weaken (α) above to $\text{tp}(\bar{b}_{m,\ell}^n, \cup \{\bar{a}_k : k < n^*, k \neq n\}) = \text{tp}(\bar{a}_n, \cup \{\bar{a}_k : k < n^*, k \neq n\})$ so without loss of generality $m_n = 1$.

3) Above for some finite $\Delta^+ \subseteq \mathbb{L}(\tau_T)$, we can in (α) demand only Δ^+ -indiscernible; also without loss of generality $\varphi_n(\bar{x}, \bar{y}_n) = \bigvee_{m < m_n} \psi_m^n(\bar{x}, \bar{y}_n)$.

Proof. [Close to 5.9.] Note

\otimes if $\bar{c} \in {}^{\ell g(\bar{x})}(\mathfrak{C})$ and $n < n^*$ and $\models \varphi_n(\bar{c}, \bar{a}_n)$ then we can find $\bar{c}' \in {}^{\ell g(\bar{x})}(\mathfrak{C})$ we have

(i) $\text{tp}(\bar{c}', \cup \{\bar{a}_k : k < n^*, k \neq n\}) = \text{tp}(\bar{c}, \cup \{\bar{a}_k : k < n^*, k \neq n\})$

(ii) $\text{tp}_{\Delta}(\bar{c}, \bar{a}_n) \neq \text{tp}_{\Delta}(\bar{c}', \bar{a}_n)$.

[Why \otimes holds? Clearly it is enough to find \bar{b}'_n such that

(i) \bar{b}_n, \bar{b}'_n realizing the same type over $\cup \{\bar{a}_k : k < n^*, k \neq n\}$

(ii) for some $m < m_n$ we have $\psi_m^n(\bar{c}, \bar{a}_n) \wedge \neg \psi_m^n(\bar{c}, \bar{a}'_n)$.]

[Why does \bar{b}'_n exist? As $\models \varphi_n[\bar{c}, \bar{a}_n]$ by (ε) for some $m < m_n$, $\models \psi_m^n[\bar{c}, \bar{a}_n]$ and by (α) + (γ), for some $\ell < \omega$, $\bar{b}'_n = \bar{b}_{m,\ell}^n$ is as required.]

By repeated use of \otimes we get $m_\ell^* < m_\ell$ such that $\langle \psi_{m_\ell^*}^n(\bar{x}, \bar{a}_n) : n < n^* \rangle$ is independent but $\psi_{m_\ell^*}^n(\bar{x}, \bar{y}_n) \in \Delta$ is finite, so n^* as required exists.

5.23 Claim. *Assume*

- (a) $\langle \bar{b}_n : n < \omega \rangle$ is indiscernible over M
- (b) $\{\varphi(\bar{x}, \bar{b}_n) : n < \omega\}$ is contradictory
- (c) $M \prec N, p \in \mathcal{S}(N), \varphi(\bar{x}, \bar{b}_0) \in p$ and $\neg\varphi(x, \bar{b}_n) \in p$ for $n > \omega$
- (d) N is $\|M\|^+$ -saturated.

Then for some $\langle b'_n : n < \omega \rangle$ we have

- (α) $\langle \bar{b}'_n : n < \omega \rangle$ is indiscernible over M such that based on $M, \bar{b}'_n \subseteq N$
- (β) $\bar{b}'_0 \in \{\bar{b}_0, \bar{b}_1\}$
- (γ) $\varphi(\bar{x}, \bar{b}'_0) \equiv \neg\varphi(\bar{x}, \bar{b}'_1)$ belongs to p .

5.24 Definition. 1) For $p \in \mathcal{S}^m(M)$ let $\mathcal{E}(p)$ be the set of pairs $(\varphi(\bar{x}, \bar{y}), \mathbf{e})$ such that

- (a) \mathbf{e} is a definable equivalence relation on ${}^{\ell g(\bar{y})}M$ in M
- (b) if $\bar{b}_1 \mathbf{e} \bar{b}_2$ then $\varphi(\bar{x}, \bar{b}_1) \in p \Leftrightarrow \varphi(\bar{x}, \bar{b}_2) \in p$.

2) $\mathcal{E}'_{\text{tp}}(p)$ is defined similarly by \mathbf{e} is definable by types.

5.25 Claim. *Assume $\varphi = \varphi(x, y), M \prec N, N$ is $\|M\|^+$ -saturated and $p \in \mathcal{S}(N)$. Then we cannot find $\{D_i : i < n_\varphi\}$, a set of ultrafilters over (N) pairwise orthogonal with $p_i = \text{Av}(M, D_i)$ such that $p \cap \{\varphi(x, \bar{b}) : b \in p_i(M)\}$ is not definable for $i < n_\varphi$.*

Now we deal with Orthogonality.

5.26 Definition. 1) Assume $\bar{\mathbf{b}}_1, \bar{\mathbf{b}}_2$ are endless indiscernible sequences. We say $\bar{\mathbf{b}}_1, \bar{\mathbf{b}}_2$ are orthogonal and write $\bar{\mathbf{b}}_1 \perp \bar{\mathbf{b}}_2$ if:

for every set A which includes $\bar{\mathbf{b}}_1 \cup \bar{\mathbf{b}}_2$, $\text{Av}(A, \bar{\mathbf{b}}_1), \text{Av}(A, \bar{\mathbf{b}}_2)$ are weakly orthogonal (see below)

2) Two complete types $p(\bar{x}), q(\bar{y})$ over A or weakly orthogonal if $p(\bar{x}) \cup q(\bar{y})$ is a complete type over A .

3) $\bar{\mathbf{b}}_1$ is strongly orthogonal to $\bar{\mathbf{b}}_2, \mathbf{b}_1 \perp_{\text{St}} \mathbf{b}_2$ iff it is orthogonal to every endless indiscernible sequence $\bar{\mathbf{b}}_2$ of finite distance from $\bar{\mathbf{b}}_2$ (see [Sh 715]).

4) An endless indiscernible sequence $\bar{\mathbf{b}}_1$ is orthogonal to $\varphi(x, \bar{a})$ iff

5) $\bar{\mathbf{b}}$ is based on A if $\bar{\mathbf{b}}$ is an indiscernible sequence and $C_A(\bar{\mathbf{b}})$ (see [Sh 715] or [Sh:93]) has boundedly many conjugations over A .

6) If $\bar{\mathbf{b}} \perp \bar{\mathbf{b}}_2$ and $\bar{\mathbf{b}}'_\ell$ is a nb to $\bar{\mathbf{b}}_\ell$ then $\bar{\mathbf{b}}'_1$ is strongly orthogonal to $\bar{\mathbf{b}}'_2$.

- 5.27 Claim.** 1) *Orthogonality is symmetric relation.*
 2) *If $\mathbf{b}_1, \mathbf{b}_2$ are orthogonal then they are perpendicular.*
 3) *No \mathbf{b} is non orthogonal to $|T|^+$ pairwise orthogonal $\langle \bar{\mathbf{b}}_i : i < |T|^+ \rangle$.*

Example: In $\text{Th}(\mathbb{R}, <)$, different initial segments are orthogonal, even two disjoint intervals. For $(\mathbb{R}, 0, 1, +, \times)$ the situation is different: any two non trivial intervals are “the same”.

- 5.28 Claim.** 1) *Assume $\lambda = \lambda^{<\lambda}$, I is a dense linear order with neither first nor last element. $\bar{\mathbf{b}} = \langle \bar{b}_t : t \in I \rangle$ an indiscernible sequence. If $|I| = \lambda$, then there is $M \supseteq \bar{\mathbf{b}}$ which is λ -saturated and λ -atomic over $\bar{\mathbf{b}}$.*
 2) *If $p \in \mathcal{S}^m(\cup \bar{\mathbf{b}})$ is λ -isolated then it is $|T|^+$ -isolated.*
 3) *M is μ -minimal (i.e. no $N, \bar{\mathbf{b}} \subseteq N \prec M, N \neq M, N$ is μ^+ -saturated) iff there is an indiscernible sequence.*

5.29 Question: If $\text{Av}(M, \bar{\mathbf{b}}_1), \text{Av}(M, \bar{\mathbf{b}}_2)$ (or with D 's) are weakly orthogonal and are perpendicular, then they are orthogonal.

5.30 Question: On say $\text{Av}(\bar{\mathbf{b}}_1, \bar{\mathbf{b}})$, $\bar{\mathbf{b}}$ endless indiscernible sequence, can we define a dependence relation exhausting the amount of indiscernible sets like dependence?

Question: For each of the following conditions can we characterize the dependent theories which satisfy it?

- (a) for any two non-trivial indiscernible sequences $\bar{\mathbf{b}}_1, \bar{\mathbf{b}}_2$, we can find $\bar{\mathbf{b}}'_\ell$ of finite distance from $\bar{\mathbf{b}}_\ell$ (see [Sh 715], for $\ell = 1, 2$) such that $\bar{\mathbf{b}}'_1, \bar{\mathbf{b}}'_2$ are not orthogonal
- (b) any two non-trivial indiscernible sequences of singletons has finite distance?
- (c) T is $\text{Th}(\mathbb{F}), \mathbb{F}$ a field (so this class includes the p -adics various reasonable fields with valuations and close under finite extensions.

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